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AN ASPECT OF LOCAL PROPERTY FOR CONVERGENCE OF ULTRASPHERICAL SERIES

By

D. P. Gupta

(*Received December* 7, 1958)

1. The ultraspherical series associated with a function $f(x) \epsilon L(-1, 1)$ is defined as

(1.1) $$\Sigma A_n P_n^{(\lambda)}(x),$$

where

(1.2) $$A_n = \frac{(n+\lambda)\,\Gamma(\lambda)}{\Gamma(\frac{1}{2}+\lambda)\,\Gamma(\frac{1}{2})} \cdot \frac{\Gamma(n+1)\,\Gamma(2\lambda)}{\Gamma(n+2\lambda)} \int_{-1}^{1} f(t)\,(1-t^2)^{\lambda-1/2} P_n^{(\lambda)}(t)\,dt,$$

$P_n^{(\lambda)}(x)$ being the nth Gegenbauer's polynomial of order λ, viz., the coefficient of r^n in the expansion of

$$(1-2rx+r^2)^{-\lambda}.$$

The Legendre series

(1.3) $$\Sigma a_n P_n(x)$$

is the particular case of the series (1.1) for the value 1/2 of the parameter λ. Under the condition

$$a_n = o\,(n^{1/2}),$$

Young[1] showed that the convergence of the series (1.3) at an internal point of the interval $(-1, +1)$ depends only on the values of $f(x)$ in the immediate neighbourhood of the point considered.

In the present paper an extension of Young's result to the case of the series (1.1) has been proved in the form of the following theorem.

Theorem. *If*

(1.4) $$A_n = o\,(n^{1-\lambda}),$$

then the convergence of the series (1.1) *at an internal point x depends only on the behaviour of the generating function $f(x)$ in the immediate neighbourhood of the point.*

2. For proving the theorem we shall require the following lemmas.

Lemma 1.[2] *For* $0 \leqslant \theta \leqslant \pi, \lambda > 0, n = 0, 1, \ldots,$

(2.1) $$|P_n^{(\lambda)}(\cos\theta)| \leqslant 2(\sin\theta)^{-\lambda} A_n^{(\lambda-1)}$$

---

1. Young (3).
2. Kogbetliantz (1).

and for $\frac{\pi}{n+1} \leqslant \theta \leqslant \pi - \frac{\pi}{n+1}$, $\lambda > 0$,

$$P_n^{(\lambda)}(\cos\theta) = \frac{2A_n^{(\lambda-1)}\cos[(n+\lambda)\theta - \lambda\pi/2]}{(2\sin\theta)^{\lambda}} + \frac{K}{(n+1)^{2-\lambda}(\sin\theta)^{\lambda+1}}, \tag{2.2}$$

where

$$A_n^{(q)} = \binom{n+q}{q} \sim n^q,$$

and K denotes a constant.

LEMMA 2.[1] *For integral n and* $\lambda > 0$,

$$P_{n+1}^{(\lambda)}(t) = \frac{(n+2\lambda)P_n^{(\lambda)}(t) + (n+2)P_{n+2}^{(\lambda)}(t)}{2(n+\lambda+1)t}. \tag{2.3}$$

3. *Proof of the theorem.* The nth partial sum of the series (1.1)

$$S_n(x) = \sum_{\nu=0}^{n} \frac{\Gamma(\lambda)\Gamma(2\lambda)}{\Gamma(\frac{1}{2}+\lambda)\Gamma(\frac{1}{2})} \cdot \frac{(\nu+\lambda)\Gamma(\nu+1)}{\Gamma(\nu+2\lambda)} \int_{-1}^{1} (1-t^2)^{\lambda-1/2} f(t) \times P_\nu^{(\lambda)}(x) P_\nu^{(\lambda)}(t)\, dt$$

$$= \frac{\Gamma(\lambda)\Gamma(2\lambda)\Gamma(n+2)}{2\Gamma(\frac{1}{2}+\lambda)\Gamma(\frac{1}{2})\Gamma(n+2\lambda)} \int_{-1}^{1} f(t)(1-t^2)^{\lambda-1/2} \times \frac{P_{n+1}^{(\lambda)}(x)P_n^{(\lambda)}(t) - P_{n+1}^{(\lambda)}(t)P_n^{(\lambda)}(x)}{x-t}\, dt$$

$$= K_n\left[\int_{-1}^{-1+\epsilon} + \int_{-1+\epsilon}^{x-\epsilon'} + \int_{x-\epsilon'}^{x+\epsilon'} + \int_{x+\epsilon'}^{1-\epsilon} + \int_{1-\epsilon}^{1}\right]$$

$$= J_1 + J_2 + J_3 + J_4 + J_5, \text{ say,} \tag{3.1}$$

having taken $\epsilon < (x+1-\epsilon', 1-x-\epsilon')$,

and
$$K_n = \frac{\Gamma(\lambda)\Gamma(2\lambda)\Gamma(n+2)}{2\Gamma(\frac{1}{2}+\lambda)\Gamma(\frac{1}{2})\Gamma(n+2\lambda)}.$$

Now

$$J_2 = K_n \int_{-1+\epsilon}^{x-\epsilon'} \frac{f(t)}{x-t}(1-t^2)^{\lambda-1/2}\left\{P_{n+1}^{(\lambda)}(x)P_n^{(\lambda)}(t) - P_{n+1}^{(\lambda)}(t)P_n^{(\lambda)}(x)\right\}dt$$

$$= J_{2,1} - J_{2,2}, \text{ say.}$$

Putting $t = \cos\theta$ and denoting $\left\{(n+\lambda)\theta - \frac{\lambda\pi}{2}\right\}$ by w_n, we have

$$J_{2,1} = K_n \int_{-1+\epsilon}^{x-\epsilon'} \frac{f(t)}{x-t}(1-t^2)^{\lambda-1/2} P_{n+1}^{(\lambda)}(x) P_n^{(\lambda)}(t)\, dt$$

$$= O(n^{2-2\lambda}) P_{n+1}^{(\lambda)}(x) \int_{-1+\epsilon}^{x-\epsilon'} \frac{f(t)}{x-t}(\sin\theta)^{2\lambda-1}$$

1. Szegö (2), p. 64.

$$\times\left\{n^{\lambda-1}(\sin\theta)^{-\lambda}\cos\omega_n+K(n+1)^{\lambda-2}(\sin\theta)^{-\lambda-1}\right\}dt$$

(by Lemma 1)

$$=O\left\{n^{1-\lambda}P_{n+1}^{(\lambda)}(x)\right\}n^{1-\lambda}\int_{-1+\epsilon}^{x-\epsilon'}\frac{f(t)}{x-t}(\sin\theta)^{\lambda-1}n^{\lambda-2}$$

$$\times\left\{n\cos\omega_n+K(\sin\theta)^{-1}\right\}dt$$

$$=O(1)\cdot\frac{1}{n}\int_{-1+\epsilon}^{x-\epsilon'}\frac{f(t)}{x-t}(\sin\theta)^{\lambda-1}\left[n\cos\omega_n+K(\sin\theta)^{-1}\right]dt$$

$$=J_{2,1,1}+J_{2,1,2}\,,\quad\text{say.}$$

By the Riemann-Lebesgue theorem

$$J_{2,1,1}=o(1),$$

and $J_{2,1,2}$ is also easily seen to be $o(1)$. Treating $J_{2,2}$ in the same manner as $J_{2,1}$, we find that

(3.2) $$J_2=o(1).$$

J_4 can also be shown to be $o(1)$ in exactly the same manner as J_2

Now

$$J_5=K_n\int_{1-\epsilon}^{1}\frac{f(t)(1-t^2)^{\lambda-1/2}}{x-t}\left\{P_{n+1}^{(\lambda)}(x)P_n^{(\lambda)}(t)-P_{n+1}^{(\lambda)}(t)P_n^{(\lambda)}(x)\right\}dt$$

$$=J_{5,1}+J_{5,2},\ \text{say.}$$

Here

$$J_{5,1}=K_n\int_{1-\epsilon}^{1}\frac{f(t)(1-t^2)^{\lambda-1/2}}{x-t}P_{n+1}^{(\lambda)}(x)P_n^{(\lambda)}(t)\,dt$$

$$=O\left\{(n+1)^{1-\lambda}P_{n+1}^{(\lambda)}(x)\right\}n^{1-\lambda}$$

$$\times O\left\{\int_{1-\epsilon}^{1}\frac{f(t)}{x-t}(1-t^2)^{\lambda-1/2}.P_n^{(\lambda)}(t)\,dt\right\}$$

(3.3) $$=O\left[\int_{1-\epsilon}^{1}\frac{d\phi_n(t)}{x-t}\right],$$

where

$$\phi_n(t)\equiv n^{1-\lambda}\int_{1-\epsilon}^{t}f(u)(1-u^2)^{\lambda-1/2}P_n^{(\lambda)}(u)\,du.$$

We first take $f(t)$ to be an even function and suppose n also to be an even integer equal to $2n_1$. Evidently then

$$f(t)(1-t^2)^{\lambda-1/2}P_n^{(\lambda)}(t)$$

is an even function. But, by hypothesis (1.4),

$$\frac{\Gamma(\lambda)\,\Gamma(2\lambda)}{\Gamma(\frac{1}{2}+\lambda)\,\Gamma(\frac{1}{2})}(n+\lambda)\frac{\Gamma(n+1)}{\Gamma(n+2\lambda)}\int_{-1}^{1} f(t)\,P_n^{(\lambda)}(t)\,(1-t^2)^{\lambda-1/2}\,dt$$

$$=o(n^{1-\lambda}),$$

or

$$n^{1-\lambda}\int_{-1}^{1} f(t)\,P_n^{(\lambda)}(t)\,(1-t^2)^{\lambda-1/2}\,dt=o(1),$$

which yields

$$n^{1-\lambda}\int_{0}^{1} f(t)\,P_n^{(\lambda)}(t)\,(1-t^2)^{\lambda-1/2}\,dt=o(1). \tag{3.4}$$

Also, using (2.2),

$$n^{1-\lambda}\int_{0}^{1-\epsilon} f(t)\,P_n^{(\lambda)}(t)\,(1-t^2)^{\lambda-1/2}\,dt$$

$$=O\left[\int_{0}^{1-\epsilon} f(t)\cos\omega_n\,dt\right]+O\left[\frac{1}{n}\int_{0}^{1-\epsilon} f(t)\,dt\right]$$

$$=o(1), \tag{3.5}$$

as $n\to\infty$.

From (3.4) and (3.5) it is clear that

$$\phi_n(1)\equiv n^{1-\lambda}\int_{1-\epsilon}^{1} f(t)\,(1-t^2)^{\lambda-1/2}\,P_n^{(\lambda)}(t)\,dt \tag{3.6}$$

$$=o(1).$$

But, by Lemma 2,

$$(n+1)^{1-\lambda}P_{n+1}^{(\lambda)}(t)=O\{n^{1-\lambda}P_n^{(\lambda)}(t)/t\}+O\{(n+2)^{1-\lambda}P_{n+2}^{(\lambda)}(t)/t\}.$$

Thus we see that, in order to use $\phi_{n+1}(1)$ in place of $\phi_n(1)$ we merely have to discuss

$$n^{1-\lambda}\int_{1-\epsilon}^{1}\frac{f(t)}{t}P_n^{(\lambda)}(t)\,(1-t^2)^{\lambda-1/2}\,dt.$$

Now, from (2.1), we get

$$|\phi_n(t)|=\left|n^{1-\lambda}\int_{1-\epsilon}^{t} f(u)\,(1-u^2)^{\lambda-1/2}P_n^{(\lambda)}(u)\,du\right|$$

$$\leqslant\int_{1-\epsilon}^{t}|f(u)|\,(1-u^2)^{(\lambda-1)/2}\,du$$

$$\leqslant c_1\,(1-t^2)^{(\lambda-1)/2}\int_{0}^{1}|f(t)|\,dt$$

(3.7) $$\leqslant c_2 (1-t^2)^{(\lambda-1)/2},$$

c_1 and c_2 being constants. Consequently

$$n^{1-\lambda}\int_{1-\epsilon}^{1} f(t)P_n^{(\lambda)}(t)\,(1-t^2)^{\lambda-1/2}\frac{dt}{t}$$

$$=\left[\phi_n(t)\frac{1}{t}\right]_{1-\epsilon}^{1}+\int_{1-\epsilon}^{1}\frac{\phi_n(t)}{t^2}\,dt$$

$$=\phi_n(1)+\int_{1-\epsilon}^{1}\frac{\phi_n(t)}{t^2}\,dt.$$

Hence

$$|\phi_n(1)-n^{1-\lambda}\int_{1-\epsilon}^{1}\frac{f(t)}{t}P_n^{(\lambda)}(t)\,(1-t^2)^{\lambda-1/2}dt\,|$$

$$=|\int_{1-\epsilon}^{1}\frac{\phi_n(t)}{t^2}\,dt\,|.$$

In view of (3.7), the right-hand side of the above may be made as small as we please, provided we choose ϵ to be sufficiently small.
So, (2.3) gives us

$$\phi_{n+1}(1)\equiv(n+1)^{1-\lambda}\int_{1-\epsilon}^{1} f(t)\,(1-t^2)^{\lambda-1/2}P_{n+1}^{(\lambda)}(t)\,dt$$
$$=O\{\phi_n(1)+\phi_{n+2}(1)+2\eta\},$$

where η is as small as we like, for we can choose ϵ after η is fixed. Thus, for all n, even or odd,

(3.8) $$\lim_{n\to\infty}|\phi_n(1)|<\eta,\ (\epsilon<\epsilon_\eta).$$

The proof given above will hold good even for an odd $f(t)$, provided we start by taking n also to be an odd number. Thus (3.8) has been proved for any odd or even function.

By similar arguments we can show that

$$\lim_{n\to\infty}|n^{1-\lambda}\int_{1-\epsilon}^{1} f(t)\,(1-t^2)^{\lambda-1/2}P_n^{(\lambda)}(t)\frac{dt}{x-t}\,|<\eta.$$

Thus

$$J_{5,1}=o(1).$$

Similarly

$$J_{5,2}=o(1).$$

J_1 can also be dealt with like J_5, and finally therefore we get

$$\lim_{n\to\infty}\left[S_n(x)-K_n\int_{x-\epsilon'}^{x+\epsilon'}\frac{f(t)\,(1-t^2)^{\lambda-1/2}}{x-t}\right.$$

$$\left.\times\left\{P_{n+1}^{(\lambda)}(x)P_n^{(\lambda)}(t)-P_{n+1}^{(\lambda)}(t)\,P_n^{(\lambda)}(x)\right\}dt\right]$$
$$=0,$$

ϵ' being any positive number, howsoever small, but less than $x+1$ and $1-x$.

This completes the proof of the theorem.

I wish to take this opportunity of acknowledging my deep gratitude to Prof. B. N. Prasad for his kind help and valuable suggestions during the preparation of this paper.

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Saugar University
Sagar
India

TAUBERIAN THEOREMS FOR ABSOLUTE RIESZ SUMMABILITY

By

T. PATI

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1.1. Let Σa_n be a given infinite series, and $\{\lambda_n\}$ a positive, monotonic increasing sequence, tending to infinity with n. We write

$$A_\lambda(\omega)=A_\lambda^0(\omega)=\sum_{\lambda_n\leqslant\omega} a_n,$$

and, for $r>0$,

$$A_\lambda^r(\omega)=\sum_{\lambda_n\leqslant\omega}(\omega-\lambda_n)^r a_n=r\int_0^\omega(\omega-\tau)^{r-1}A_\lambda(\tau)\,d\tau.$$

The series Σa_n is said to be absolutely summable (R, λ, r), or summable $|R, \lambda, r|$, $r\geqslant 0$, if

$$A_\lambda^r(\omega)/\omega^r \in BV(A, \infty),^1$$

where A is some finite positive number.[2]

1.2. The only general results known to us so far in the theory of Tauberian theorems for absolute Riesz summability of infinite series are the following theorems, due to the present author and Bhatt respectively.

THEOREM A.[3] If (i) Σa_n is summable $|R, \lambda, k|$, $k>0$, (ii) $\{a_n\lambda_n/\mu_n\}\in BV$[4] $(\mu_n=\lambda_n-\lambda_{n-1})$ and (iii) $\{\lambda_n/\lambda_{n+1}\}\in BV$, then Σa_n is absolutely convergent.

THEOREM B.[5] If (i) Σa_n is summable $|R, \lambda, k|$, $k>0$,

$$\text{(ii)}\quad\left\{\frac{1}{\lambda_n}\sum_1^n\lambda_\nu a_\nu\right\}\in BV$$

and (iii) $\{\lambda_n/\lambda_{n+1}\}\in BV$, then Σa_n is absolutely convergent.

Condition (ii) of Theorem B is less stringent than condition (ii) of Theorem A. In fact, with $\mu_1=\lambda_1$, $\mu_n=\lambda_n-\lambda_{n-1}$ $(n=2, 3, \ldots)$,

---

1. By '$f(x)\in BV(h, k)$' we mean that $f(x)$ is a function of bounded variation over (h, k). When the variable with respect to which the function is of bounded variation is emphasised, BV is followed by it as the suffix, as in '$f(x, t)\in BV_x(h, k)$'

2. Obrechkoff (3), (4) Throughout A denotes a finite positive number, not necessarily the same at each occurrence.

3. Pati (5).

4. By '$\{c_n\}\in BV$' we mean that $\Sigma_n|c_n-c_{n-1}|<\infty$.

5. Bhatt (1).

$$c_n = a_n\lambda_n/\mu_n, \; \Delta c_n = c_n - c_{n+1},$$

we have

$$\frac{1}{\lambda_n}\sum_1^n \lambda_\nu a_\nu = \frac{1}{\lambda_n}\sum_1^n c_\nu \mu_\nu = \frac{1}{\lambda_n}\sum_1^{n-1} \lambda_\nu \Delta c_\nu + c_n$$

$$= -\frac{1}{\lambda_n}\sum_1^{n-1} (\lambda_n - \lambda_\nu)\Delta c_\nu + c_1.$$

Hence, condition (ii) of Theorem B is the discrete analogue of the summability $| R, \lambda, 1 |$ of the sequence $\{c_n\}$, and is less stringent than condition (ii) of Theorem A.

Both these theorems are of the 'sufficiency type', and it may be noted that, among the conditions involving the sequence $\{\lambda_n\}$, the one that is common to both is the condition (iii), namely, that

$$\{\lambda_n/\lambda_{n+1}\} \in BV.$$

We observe that the sequence $\{\lambda_n/\lambda_{n+1}\}$ is generated by the greatest lower bounds of the function

$$F(\omega)/\omega$$

over the intervals $(\lambda_n, \lambda_{n+1})$ for $n=1, 2, 3, \ldots$, where $F(\omega)$ is the step-function, defined as

$$F(\omega)=0 \text{ for } \omega<\lambda_1 \,;\, F(\omega)=\lambda_n \text{ for } \lambda_n \leqslant \omega < \lambda_{n+1}.$$

It is easily seen that the bounded variation of $F(\omega)/\omega$ over (A, ∞) is equivalent to the convergence of the series

$$\Sigma\left(1-\frac{\lambda_{n-1}}{\lambda_n}\right) = \Sigma(\mu_n/\lambda_n),$$

and it follows from this that if $F(\omega)/\omega \in BV\,(A, \infty)$, then $\{\lambda_n/\lambda_{n+1}\} \in BV$. But the converse is not necessarily true, as is seen by taking $\lambda_n = n$.

The object of the present paper is to obtain necessary and sufficient conditions for the effectiveness of certain Tauberian conditions involving the sequence $\{c_n\} \equiv \{a_n \lambda_n/\mu_n\}$ for absolutely Riesz summable series. We find that these necessary and sufficient conditions involve the functions

$$F(\omega)/\omega \quad \text{or} \quad \frac{1}{\omega^2}\int_0^\omega F(t)\,dt,$$

according as we use the Tauberian condition for transition from summability $| R, \lambda, 1 |$ to absolute convergence or from summability $| R, \lambda, 2 |$ to summability $| R, \lambda, 1 |$.

1.3. We adopt the following notations:

$$\mu_1=\lambda_1, \mu_n=\lambda_n-\lambda_{n-1} \ (n=2,3,\dots) \ ; \ c_n=a_n\lambda_n/\mu_n \ ;$$

$$D(t)=\sum_{\lambda_n\leqslant t}\Delta c_n, \ \Delta c_n=c_n-c_{n+1}.$$

$D^1(t)$ and $D^2(t)$ denote the Riesz sums of type λ and orders 1 and 2, respectively, of the series $\Sigma\Delta c_n$. We also write

$$C(t)=\sum_{\lambda_n\leqslant t}\mu_n c_n, \ \bar{D}(t)=\sum_{\lambda_n\leqslant t}\lambda_n\Delta c_n.$$

2.1. We establish the following two theorems:

THEOREM 1. *A necessary and sufficient condition that Σa_n be absolutely convergent whenever it is summable $|R,\lambda,1|$ and $\{c_n\}\in BV$, that is, $D(\omega)\in BV$ (A,∞), is that*

$$\Big(D(\omega)-a_1\Big)\frac{F(\omega)}{\omega}\in BV(A,\infty).$$

THEOREM 2. *A necessary and sufficient condition that Σa_n be summable $|R,\lambda,1|$ whenever it is summable $|R,\lambda,2|$ and the sequence $\{c_n\}$ is summable $|R,\lambda,1|$, that is, $\omega^{-1}D^1(\omega)\in BV(A,\infty)$, is that*

$$\left(\frac{D^1(\omega)}{\omega}-a_1\right)\frac{1}{\omega^2}\int_0^\omega F(t)\,dt\in BV(A,\infty).$$

2.2. We shall require the following lemmas for the proof of the theorems.

LEMMA 1.[1] *If Σa_n is summable $|R,\lambda,k|$, $k\geqslant 0$, then it is summable $|R,\lambda,k'|$ for every $k'>k$.*

LEMMA 2.[2] *If $G(t)\in BV(0,\infty)$, and uniformly in $\tau\geqslant 0$*

$$\int_\tau^x \phi(x,t)\,dt\in BV_x(\tau,\infty),$$

then

$$\int_0^x G(t)\phi(x,t)\,dt\in BV_x(0,\infty).$$

LEMMA 3.[3] *If $\chi(t)\in BV(0,\infty)$, and*

$$\iota(x)=\int_0^\infty \phi(x,t)\,d\chi(t)$$

converges for every $x>0$, and if uniformly in $t\geqslant 0$

1. Obrechkoff (3); this is the so-called 'first theorem of consistency' for absolute Riesz summability of infinite series.

2. Pati (5), Lemma 3 of Chapter II.

3. Knopp and Vanderburg (2).

$$\phi(x,t) \in BV_x(0, \infty),$$

then

$$c(x) \in BV(0, \infty).$$

LEMMA 4. *For* $t>0$

$$C(t)=\bar{D}(t)-F(t)\{D(t)-a_1\}.$$

Proof. For $\lambda_m \leqslant t < \lambda_{m+1}$,

$$C(t)=\sum_{\lambda_n \leqslant t} \mu_n c_n = \sum_1^m \mu_n c_n$$

$$=\sum_1^{m-1} \lambda_n \Delta c_n + \lambda_m c_m$$

$$=\sum_1^{m} \lambda_n \Delta c_n + \lambda_m c_{m+1}$$

$$=\sum_{\lambda_n \leqslant t} \lambda_n \Delta c_n + \lambda_m c_{m+1}$$

$$=\bar{D}(t)-F(t)\{D(t)-a_1\},$$

by definition.

3.1. *Proof of Theorem* 1. We assume, as we may without any loss of generality, that $A=\lambda_1$. We prove the necessity and the sufficiency of the condition separately.

Sufficiency. We have

$$A_\lambda(\omega)-A_\lambda^1(\omega)/\omega$$

$$=\omega^{-1} \sum_{\lambda_n \leqslant \omega} \mu_n c_n = \omega^{-1} C(\omega).$$

By Lemma 4,

$$\omega^{-1}C(\omega)=\omega^{-1}\bar{D}(\omega)-\omega^{-1}F(\omega)\{D(\omega)-a_1\}$$

$$=D(\omega)-\omega^{-1}D^1(\omega)-\omega^{-1}F(\omega)\{D(\omega)-a_1\}.$$

By hypothesis

$$D(\omega) \in BV(0, \infty),$$

and, therefore, by Lemma 1,

$$\omega^{-1}D^1(\omega) \in BV(0, \infty).$$

Also, since

$$\omega^{-1}F(\omega)\{D(\omega)-a_1\} \in BV(0, \infty),$$

we have

$$\omega^{-1}C(\omega) \in BV(0, \infty),$$

as required.

Necessity. By hypothesis

$$A_\lambda(\omega)-\omega^{-1}A_\lambda^1(\omega)\in BV(0,\infty),$$

that is, by the analysis of the proof of the sufficiency-part,

$$D(\omega)-\omega^{-1}D^1(\omega)-\omega^{-1}F(\omega)\{D(\omega)-a_1\}\in BV(0,\infty),$$

and, therefore, by hypothesis and Lemma 1, ultimately

$$\omega^{-1}F(\omega)\{D(\omega)-a_1\}\in BV(0,\infty).$$

3.2. *Proof of Theorem 2.* As in the proof of Theorem 1, we assume, as we may, that $A=\lambda_1$.

Sufficiency. We have

$$\begin{aligned}
&\omega^{-1}A_\lambda^1(\omega)-\omega^{-2}A_\lambda^2(\omega)\\
&=\omega^{-2}\int_0^\omega C_\lambda(t)\,dt\\
&=\omega^{-2}\int_0^\omega\Big\{\bar{D}_1(t)-F(t)\{D(t)-a_1\}\Big\}\,dt\\
&=\omega^{-2}\sum_{\lambda_n\leqslant\omega}(\omega-\lambda_n)\lambda_n\Delta c_n-\omega^{-2}\int_0^\omega F(t)\{D(t)-a_1\}\,dt\\
&=\omega^{-1}D^1(\omega)-\omega^{-2}D^2(\omega)-\omega^{-2}\int_0^\omega F(t)\{D(t)-a_1\}\,dt.
\end{aligned}$$

Since, by hypothesis and Lemma 1,

$$\omega^{-1}D^1(\omega)\in BV(0,\infty),$$

and

$$\omega^{-2}D^2(\omega)\in BV(0,\infty),$$

we have only to show that

$$I(\omega)=\omega^{-2}\int_0^\omega F(t)\{D(t)-a_1\}\,dt\in BV(0,\infty).$$

Now

$$\begin{aligned}
I(\omega)=\omega^{-2}\int_0^\omega tF(t)\,d\Big(t^{-1}D^1(t)\Big)&\\
+\omega^{-2}\int_0^\omega F(t)\Big(t^{-1}D^1(t)-a_1\Big)\,dt&\\
=I_1(\omega)+I_2(\omega),\text{ say.}&
\end{aligned}$$

Since, uniformly in $t>0$,

$$\omega^{-2} t F(t) \in BV_{\omega}(t, \infty),$$

and, by hypothesis,

$$t^{-1} D^{1}(t) \in BV(0, \infty),$$

it follows, by Lemma 3, that

$$I_1(\omega) \in BV(0, \infty).$$

Next, by integration by parts,

$$I_2(\omega) = \left(\omega^{-1} D^{1}(\omega) - a_1\right) . \omega^{-2} \int_0^{\omega} F(t)\, dt$$
$$- \omega^{-2} \int_0^{\omega} \left(\int_0^{t} F(\tau)\, d\tau\right) d\left(t^{-1} D^{1}(t)\right)$$

$$= I_{21}(\omega) - I_{22}(\omega), \text{ say.}$$

Since

$$t^{-1} D^{1}(t) \in BV(0, \infty),$$

and, uniformly in $t>0$,

$$\omega^{-2} \int_0^{t} F(\tau)\, d\tau \in BV_{\omega}(t, \infty),$$

it follows, by Lemma 3, that

$$I_{22}(\omega) \in BV(0, \infty).$$

Also, by hypothesis,

$$I_{21}(\omega) \in BV(0, \infty).$$

Hence $I(\omega) \in BV(0, \infty)$, as required.

Necessity. Following the analysis of the proof of the sufficiency-part, we have, by hypothesis

$$\omega^{-1} D^{1}(\omega) - \omega^{-2} D^{2}(\omega) - \omega^{-2} \int_0^{\omega} F(t) \{D(t) - a_1\}\, dt \in BV(0, \infty),$$

and, since by hypothesis, and consistency

$$\omega^{-1} D^{1}(\omega) \in BV(0, \infty),$$

and

$$\omega^{-2} D^{2}(\omega) \in BV(0, \infty),$$

we have

$$\omega^{-2} \int_0^{\omega} F(t) \{D(t) - a_1\}\, dt \in BV(0, \infty).$$

Following the notations used in the proof of the sufficiency-part, we have by hypothesis and Lemma 3

$$I_1(\omega) \in BV(0, \infty),$$

and

$$I_{22}(\omega) \in BV(0, \infty).$$

Hence, finally,

$$\left(\omega^{-1}D^1(\omega)-a_1\right)\frac{1}{\omega^2}\int_0^\omega F(t)\,dt \in BV(0, \infty).$$

4. *Remarks.* We make a few relevant observations here. It follows from Lemma 2 that if

$$\text{(i)} \quad t^{-1}F(t) \in BV(0, \infty),$$

then

$$\text{(ii)} \quad \omega^{-2}\int_0^\omega F(t)\,dt \in BV(0, \infty).$$

As we have observed earlier (section 1.2), (i) is equivalent to the convergence of

$$\text{(iii)} \qquad \Sigma(\mu_n/\lambda_n).$$

Now, for $\lambda_m \leqslant \omega < \lambda_{m+1}$,

$$\frac{1}{\omega^2}\int_0^\omega F(t)\,dt = \frac{1}{\omega^2}\left[\sum_{n=2}^{m}\int_{\lambda_{n-1}}^{\lambda_n} F(t)\,dt + \int_{\lambda_m}^{\omega} F(t)\,dt\right]$$

$$= \frac{1}{\omega^2}\left[\sum_{n=2}^{m}\lambda_{n-1}\mu_n + \lambda_m(\omega-\lambda_m)\right]$$

$$= \frac{1}{\omega^2}\left[-\sum_{1}^{m}\lambda_n\mu_n + \sum_{2}^{m}(\lambda_n{}^2-\lambda_{n-1}{}^2) + \lambda_m(\omega-\lambda_m) + \lambda_1{}^2\right]$$

$$= \frac{1}{\omega^2}\sum_{1}^{m}(\omega-\lambda_n)\,\mu_n$$

$$= \frac{1}{\omega}\sum_{\lambda_n \leqslant \omega}(\omega-\lambda_n)\frac{\mu_n}{\lambda_n} - \frac{1}{\omega^2}\sum_{\lambda_n \leqslant \omega}(\omega-\lambda_n)^2\frac{\mu_n}{\lambda_n}$$

$$= R^1(\omega) - R^2(\omega),$$

where $R^1(\omega)$ and $R^2(\omega)$ are the Riesz means of type λ and orders 1 and 2, respectively, of the series (iii). Hence, it is but natural that (i) should

imply (ii). It would be worthwhile to study the relations between the conditions (ii) and

$$\text{(iv)} \quad \{\lambda_n/\lambda_{n+1}\} \in BV.$$

I take this opportunity of expressing my warmest thanks to Professor B. N. Prasad for his very kind encouragement.

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The University
Allahabad
India

A TAUBERIAN THEOREM FOR ABSOLUTE SUMMABILITY

By

SYED MOHAMMAD MAZHAR

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1.1. A series Σa_n is said to be summable $|A|_k$, $k \geqslant 1$, if the series $\Sigma a_n e^{-ns}$ is convergent and its sum function $f(s)$ satisfies the condition

$$\int_0^\infty (e^s-1)^{k-1} |f'(s)|^k ds < \infty^1.$$

Let $\alpha > -1$, and $\sum_0^\infty A_n^{(\alpha)} x^n = (1-x)^{-\alpha-1} \sum_0^\infty a_n x^n$;

$$\sum_0^\infty E_n^{(\alpha)} x^n = (1-x)^{-\alpha-1},\ C_n^{(\alpha)} = \frac{A_n^{(\alpha)}}{E_n^{(\alpha)}}.$$

Let $b_n = n a_n$ and let $B_n^{(\alpha)}$, $d_n^{(\alpha)}$ and $e_n^{(\alpha)}$, for $\alpha > -1$, be defined by the identities

$$\sum_0^\infty B_n^{(\alpha)} x^n = (1-x)^{-\alpha-1} \sum_0^\infty b_n x^n;$$

$$d_n^{(\alpha)} = \frac{B_n^{(\alpha)}}{E_n^{(\alpha)}};\ e_n^{(\alpha)} = \frac{B_n^{(\alpha-1)}}{E_n^{(\alpha)}}.$$

Then we have[2]

(1.1.1) $$e_n^{(\alpha)} = n \Delta C_n^{(\alpha)};$$

(1.1.2) $$e_n^{(\alpha+1)} = (\alpha+1)\,[C_n^{(\alpha)} - C_n^{(\alpha+1)}].$$

We write throughout

$$\Delta u_n = u_n - u_{n-1}.$$

The series Σa_n is said to be summable $|C, \alpha|_k$ if $\Sigma n^{k-1} |\Delta C_n^{(\alpha)}|^k < \infty$.[3]

Since the $(\alpha+1)$-th Cesàro mean of the sequence $\{b_n\}$ is given by $\frac{B_n^{(\alpha)}}{E_n^{(\alpha+1)}} = e_n^{(\alpha+1)}$, the sequence $\{b_n\}$ or the series $\Sigma \Delta b_n$ is summable $|C, \alpha+1|_k$ if $\Sigma n^{k-1} |\Delta e_n^{(\alpha+1)}|^k < \infty$, where $k \geqslant 1$ and $\alpha > -1$.

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1. Cf. Flett (1).
2. Kogbetliantz (3).
3. Flett (1).

It is easily seen that, by definition, summability $|A|_k$ and summability $|C, \alpha|_k$ are the same as summability $|A|$ and summability $|C, \alpha|$ if $k=1$.

1.2. Hyslop[1] has established the following Tauberian theorem for absolute summability.

THEOREM. *If the series Σa_n is summable $|A|$ and $\Sigma \Delta b_n$ is summable $|C, \alpha+1|$, where $\alpha \geqslant 0$, then Σa_n is summable $|C, \alpha|$.*

The object of the present paper is to generalize this theorem by introducing the extended definitions of absolute Abel and Cesàro Summability referred to above as summability $|A|_k$ and summability $|C, \alpha|_k$.

I wish to express my warmest thanks to Prof. B. N. Prasad for his kind help and advice during the preparation of this paper.

2.1. We establish the following theorem.

THEOREM. *If Σa_n is summable $|A|_k$, $k \geqslant 1$, and $\Sigma \Delta b_n$ is summable $|C, \alpha+1|_k$, where $\alpha > -1$, then Σa_n is summable $|C, \alpha|_k$.*

2.2. *Proof.* Let $m=[w]$, $w=s^{-1}$ and $J(w)=\dfrac{1}{w(1-e^{-1/w})}$, $w \geqslant 1$.

Taking two suitable constants J_1 and J_2, we have

$$J_1 < J(w) < J_2.$$

Now let

$$\psi_N = \int_1^N |e_m^{(\alpha+1)}|^k w^{-1} J(w)\, e^{(k-1)/w}\, dw$$

$$= \sum_{n=1}^{N-1} n^{-1} |e_n^{(\alpha+1)}|^k \int_n^{n+1} n\, e^{(k-1)/w}\, w^{-1} J(w)\, dw.$$

$$> \frac{J_1}{2} \sum_{n=1}^{N-1} n^{-1} |e_n^{(\alpha+1)}|^k$$

$$= \frac{J_1}{2} \sum_{n=1}^{N-1} n^{k-1} |\Delta C_n^{(\alpha+1)}|^k.$$

From (1.1 2) we have

$$\Sigma n^{k-1} |(\alpha+1) \Delta C_n^{(\alpha)}|^k \leqslant [(\Sigma n^{k-1} |(\alpha+1) \Delta C_n^{(\alpha+1)}|^k)^{1/k} + (\Sigma n^{k-1} |\Delta e_n^{(\alpha+1)}|^k)^{1/k}]^k.$$

Therefore it is sufficient to show that $\psi_N = O(1)$.

1. Hyslop (2).

Now

$$\psi_N=\int_1^N |e_m^{(\alpha+1)}\, w^{-1/k} J(w)^{1/k} e^{(k-1)/kw}+w^{-2/k}(e^{1/w}-1)^{(k-1)/k} f'(1/w)$$
$$-w^{-2/k}(e^{1/w}-1)^{(k-1)/k} f'(1/w)|^k\, dw$$
$$\leqslant \left[\left(\int_1^N |e_m^{(\alpha+1)}\, w^{-1/k} J(w)^{1/k} e^{(k-1)/kw}+w^{-2/k}(e^{1/w}-1)^{(k-1)/k} f'(1/w)|^k\, dw\right)^{1/k}\right.$$
$$\left.+\left(\int_1^N w^{-2}(e^{1/w}-1)^{k-1}|f'(1/w)|^k\, dw\right)^{1/k}\right]^k$$
$$=[\mathscr{J}_1^{1/k}+\mathscr{J}_2^{1/k}]^k, \text{ say.}$$

We have

$$\mathscr{J}_2=\int_1^N w^{-2}(e^{1/w}-1)^{k-1}|f'(1/w)|^k\, dw$$
$$=\int_{1/N}^1 (e^s-1)^{k-1}|f'(s)|^k\, ds$$
$$=O(1), \text{ by hypothesis.}$$

Since

$$f'(1/w)=-\sum_0^\infty b_n e^{-n/w}=-(1-e^{-1/w})^{\alpha+1}\sum_0^\infty E_n^{(\alpha+1)} e^{-n/w} e_n^{(\alpha+1)},$$

we have

$$\mathscr{J}_1=\int_1^N |e_m^{(\alpha+1)}\, w^{-1/k} J(w)^{1/k} e^{(k-1)/kw}+w^{-2/k}(e^{1/w}-1)^{(k-1)/k} f'(1/w)|^k\, dw$$
$$=\int_1^N |e_m^{(\alpha+1)}\, w^{-1/k} J(w)^{1/k} e^{(k-1)/kw}-w^{-2/k}(e^{1/w}-1)^{(k-1)/k}(1-e^{-1/w})^{1+\alpha}$$
$$\times\sum_0^\infty E_n^{(\alpha+1)} e_n^{(\alpha+1)} e^{-n/w}|^k\, dw$$

$$=\int_1^N |w^{-1/k} J(w)^{1/k} e^{(k-1)/kw}(1-e^{-1/w})^{\alpha+2}\sum_{n=0}^\infty E_n^{(\alpha+1)} e^{-n/w}$$
$$\times(e_m^{(\alpha+1)}-e_n^{(\alpha+1)})|^k\, dw$$

$$=\int_1^N |w^{-1/k} J(w)^{1/k} e^{(k-1)/kw}(1-e^{-1/w})^{\alpha+2}\sum_{n=0}^\infty E_n^{(\alpha+1)} e^{-n/w}$$
$$\times\sum_{r=n+1}^m \Delta e_r^{(\alpha+1)}|^k\, dw$$

$$\mathscr{I}_1=\int_1^N |w^{-1/k}\mathcal{J}(w)^{1/k} e^{(k-1)/kw}(1-e^{-1/w})^{\alpha+2}$$

$$\times\left\{\sum_{n=0}^{m}\ldots\sum_{r=n+1}^{m}-\sum_{n=m+1}^{\infty}\ldots\sum_{r=m+1}^{n}\ldots\right\}|^k dw$$

$$<\left[\left\{\int_1^N(\phi(w)\sum_{n=0}^{m} E_n^{(\alpha+1)}e^{-n/w}\sum_{r=n}^{m}|\Delta e_r^{(\alpha+1)}|)^k dw\right\}^{1/k}\right.$$

$$\left.+\left\{\int_1^N(\phi(w)\sum_{n=m+1}^{\infty} E_n^{(\alpha+1)}e^{-n/w}\sum_{r=m+1}^{n}|\Delta e_r^{(\alpha+1)}|)^k dw\right\}^{1/k}\right]^k$$

[where $\phi(w)=w^{-1/k}\mathcal{J}(w)^{1/k}e^{(k-1)/kw}(1-e^{-1/w})^{\alpha+2}$]

$$=[I_1^{(1/k)}+I_2^{(1/k)}]^k, \text{ say.}$$

Now

$$I_1=\int_1^N\{\phi(w)\sum_{n=0}^{m} E_n^{(\alpha+1)}e^{-n/w}\sum_{r=n}^{m}|\Delta e_r^{(\alpha+1)}|\}^k dw$$

$$=\int_1^N\{\phi(w)\sum_{r=1}^{m}|\Delta e_r^{(\alpha+1)}|\sum_{n=0}^{r}E_n^{(\alpha+1)}e^{-n/w}\}^k dw$$

$$=\int_1^N\{\phi(w)\sum_{r=1}^{m}|\Delta e_r^{(\alpha+1)}|+\phi(w)\sum_{r=1}^{m}|\Delta e_r^{(\alpha+1)}|\sum_{n=1}^{r}E_n^{(\alpha+1)}e^{-n/w}\}^k dw$$

$$\leqslant\left[\left\{\int_1^N\{\phi(w)\sum_{r=1}^{m}|\Delta e_r^{(\alpha+1)}|\}^k dw\right\}^{1/k}+\left\{\int_1^N(\phi(w)\sum_{r=1}^{m}|\Delta e_r^{(\alpha+1)}|\right.\right.$$

$$\left.\left.\times\sum_{n=1}^{r}E_n^{(\alpha+1)}e^{-n/w})^k dw\right\}^{1/k}\right]^k$$

$$=[L_1^{1/k}+L_2^{1/k}]^k, \text{ say.}$$

Now

$$L_1=\int_1^N(\phi(w)\sum_{r=1}^{m}|\Delta e_r^{(\alpha+1)}|)^k dw$$

$$=\int_1^N w^{-1}\mathcal{J}(w)e^{(k-1)/w}(1-e^{-1/w})^{(\alpha+2)k}(\sum_{r=1}^{m}|\Delta e_r^{(\alpha+1)}|)^k dw$$

$$L_1 \leqslant \int_1^N w^{-1} J(w)\, e^{(k-1)/w} (1-e^{-1/w})^{(\alpha+2)k} w^{k-1} \sum_{r=1}^{m} |\Delta e_r^{(\alpha+1)}|^k\, dw$$

$$= O\left\{\sum_{r=1}^{N} |\Delta e_r^{(\alpha+1)}|^k \int_r^\infty e^{(k-1)/w} w^{k-2} J(w) \{w J(w)\}^{-(\alpha+2)k}\, dw\right\}$$

$$= O\left\{\sum_{r=1}^{N} |\Delta e_r^{(\alpha+1)}|^k \int_r^\infty e^{(k-1)/w} w^{k-2} w^{-(\alpha+2)k}\, dw\right\},$$

since $\{J(w)\}^{1-(\alpha+2)k} \epsilon B$,

$$= O\left\{\sum_{r=1}^{N} |\Delta e_r^{(\alpha+1)}|^k\right\}$$

$$= O(1)$$

by hypothesis, and

$$L_2 = \int_1^N \{\phi(w)\}^k \Big(\sum_{r=1}^{m} |\Delta e_r^{(\alpha+1)}| \sum_{n=1}^{r} E_n^{(\alpha+1)} e^{-n/w}\Big)^k dw$$

$$\leqslant \int_1^N \{\phi(w)\}^k \Big(\sum_{r=1}^{m} |\Delta e_r^{(\alpha+1)}|^k \,\Big|\sum_{n=1}^{r} E_n^{(\alpha+1)} e^{-n/w}\Big|^k\Big) m^{k-1}\, dw$$

$$< \int_1^N \{\phi(w)\}^k w^{k-1} \Big(\sum_{r=1}^{m} |\Delta e_r^{(\alpha+1)}|^k \sum_{n=1}^{r} (E_n^{(\alpha+1)})^k e^{-kn/w} r^{k-1}\Big)\, dw$$

$$= \int_1^N \{\phi(w)\}^k w^{k-1} \Big(\sum_{r=1}^{m} r^{k-1} |\Delta e_r^{(\alpha+1)}|^k \sum_{n=1}^{r} (E_n^{(\alpha+1)})^k e^{-kn/w}\Big)\, dw$$

$$= O\Big[\sum_{r=1}^{N} r^{k-1} |\Delta e_r^{(\alpha+1)}|^k \sum_{n=1}^{r} (E_n^{(\alpha+1)})^k \int_r^\infty w^{k-1} \{\phi(w)\}^k e^{-kn/w}\, dw\Big]$$

$$= O\Big[\sum_{r=1}^{N} r^{k-1} |\Delta e_r^{(\alpha+1)}|^k \sum_{n=1}^{r} (E_n^{(\alpha+1)})^k \int_r^\infty w^{k-1} w^{-1} J(w) e^{(k-1)/w} \times \{w J(w)\}^{-(\alpha+2)k} e^{-kn/w}\, dw\Big].$$

Since

$$\int_r^\infty w^{k-2-(\alpha+2)k} e^{-kn/w}\, dw = \int_0^{1/r} t^{\alpha k + k} e^{-knt}\, dt$$

$$\leqslant r^{-\alpha k-k}\left\{\frac{1}{kn}\left(1-e^{-kn/r}\right)\right\}$$

$$=O(n^{-1}r^{-\alpha k-k}).$$

So

$$L_2=O\{\sum_{r=1}^{N} r^{k-1}|\Delta e_r^{(\alpha+1)}|^k \sum_{n=1}^{r}(E_n^{(\alpha+1)})^k n^{-1} r^{-\alpha k-k}\}$$

$$=O\{\sum_{r=1}^{N} r^{k-1}|\Delta e_r^{(\alpha+1)}|^k\}$$

$$=O(1).$$

Hence, finally,

$$I_1=O(1).$$

Next

$$I_2=\int_1^N\{\phi(w)\sum_{n=m+1}^{\infty}E_n^{(\alpha+1)}e^{-n/w}\sum_{r=m+1}^{n}|\Delta e_r^{(\alpha+1)}|\}^k\,dw$$

$$<\int_1^N\{\phi(w)\}^k\,(\sum_{n=m+1}^{\infty}E_n^{(\alpha+1)}e^{-n/w}\sum_{r=m+1}^{n}|\Delta e_r^{(\alpha+1)}|)^k\,dw$$

$$=\int_1^N\{\phi(w)\}^k\,(\sum_{r=m+1}^{\infty}|\Delta e_r^{(\alpha+1)}|\sum_{n=r}^{\infty}E_n^{(\alpha+1)}e^{-n/w})^k\,dw$$

$$<\int_1^N\{\phi(w)\}^k\,(\sum_{r=m+1}^{\infty}|\Delta e_r^{(\alpha+1)}|e^{-r/3w}\sum_{n=r}^{\infty}E_n^{(\alpha+1)}e^{-2n/3w})^k\,dw$$

$$\leqslant\int_1^N\{\phi(w)\}^k(\sum_{r=m+1}^{\infty}|\Delta e_r^{(\alpha+1)}|^k\,.\,|\sum_{n=r}^{\infty}E_n^{(\alpha+1)}e^{-2n/3w}|^k)$$

$$\times(\sum_{r=m+1}^{\infty}e^{-r/3w\,\cdot\,k/(k-1)})^{k-1}.$$

But

$$(\sum_{r=m+1}^{\infty}e^{-r/3w\,\cdot\,k/(k-1)})^{k-1}$$

$$=\{e^{-(m+1)k/3(k-1)w}/(1-e^{-k/3w(k-1)})\}^{(k-1)}$$

$$<\frac{1}{(1-e^{-1/3w})^{k-1}}$$

$$=\frac{(3w)^{k-1}}{\{3w(1-e^{-1/3w})\}^{(k-1)}}$$

$$=\{\mathcal{J}(3w)\}^{(k-1)}(3w)^{k-1}$$

$$=O(w^{k-1}),$$

since $\mathcal{J}(w)\in B$.

Therefore

$$I_2=O\left\{\int_1^N\{\phi(w)\}^k\left(\sum_{r=m+1}^{\infty}|\Delta e_r^{(\alpha+1)}|^k\right)w^{k-1}\left|\sum_{n=r}^{\infty}E_n^{(\alpha+1)}e^{-2n/3w}\right|^k dw\right\}$$

$$=O\left[\int_1^N\{\phi(w)\}^k w^{k-1}\sum_{r=m+1}^{\infty}|\Delta e_r^{(\alpha+1)}|^k e^{-rk/3w}\right.$$

$$\left.\times\left|\sum_{n=r}^{\infty}E_n^{(\alpha+1)}e^{-n/3w}\right|^k dw\right]$$

$$=O\left[\sum_{r=1}^{\infty}|\Delta e_r^{(\alpha+1)}|^k\int_1^r\{\phi(w)\}^k w^{k-1}e^{-rk/3w}\left(\sum_{n=0}^{\infty}E_n^{(\alpha+1)}e^{-n/3w}\right)^k dw\right]$$

$$=O\left[\sum_{r=1}^{\infty}|\Delta e_r^{(\alpha+1)}|^k\int_1^r\{\phi(w)\}^k w^{k-1}e^{-rk/3w}\cdot\frac{1}{(1-e^{-1/3w})^{(\alpha+2)k}}dw\right]$$

$$=O\left[\sum_{r=1}^{\infty}r^{k-1}|\Delta e_r^{(\alpha+1)}|^k\int_1^r w^{-1}\mathcal{J}(w)\,e^{(k-1)/w}(1-e^{-1/w})^{(\alpha+2)k}e^{-rk/3w}\right.$$

$$\left.\times\frac{1}{(1-e^{-1/3w})^{(\alpha+2)k}}dw\right]$$

$$=O\left[\sum_{r=1}^{\infty}r^{k-1}|\Delta e_r^{(\alpha+1)}|^k\int_1^r w^{-1}\mathcal{J}(w)\,e^{-rk/3w}\,dw\right],$$

$$\text{since}\quad e^{(k-1)/w}\left(\frac{1-e^{-1/w}}{1-e^{-1/3w}}\right)^{(\alpha+2)k}\in B,$$

$$=O\left\{\sum_{r=1}^{\infty}r^{k-1}|\Delta e_r^{(\alpha+1)}|^k\int_1^r w^{-2}w\mathcal{J}(w)\,e^{-rk/3w}\,dw\right\}$$

$$=O\left\{\sum_{r=1}^{\infty} r^{k-1}|\Delta e_r^{(\alpha+1)}|^k r\mathcal{J}(r)\int_1^r w^{-2}e^{-rk/3w}\,dw\right\},$$

since $w\mathcal{J}(w)$ is an increasing function of w,

$$=O\left\{\sum_{r=1}^{\infty} r^{k-1}|\Delta e_r^{(\alpha+1)}|^k r\mathcal{J}(r)\int_{1/r}^1 e^{-rkt/3}\,dt\right\}$$

$$=O\left\{\sum_{r=1}^{\infty} r^{k-1}|\Delta e_r^{(\alpha+1)}|^k r\mathcal{J}(r)\frac{(e^{-k/3}-e^{-rk/3})}{rk/3}\right\}$$

$$=O\left\{\sum_{r=1}^{\infty} r^{k-1}|\Delta e_r^{(\alpha+1)}|^k \mathcal{J}(r)(1-e^{-rk/3})\right\}$$

$$=O\left\{\sum_{r=1}^{\infty} r^{k-1}|\Delta e_r^{(\alpha+1)}|^k\right\},$$ since $\mathcal{J}(r)$ and $1-e^{-rk/3}$ are bounded,

$$=O(1)$$

by hypothesis.

Therefore

$$\mathcal{J}_1=O(1).$$

Hence finally

$$\psi_N=O(1).$$

This completes the proof of the theorem.

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The University
Allahabad
India

ON THE ABSCISSA OF ABSOLUTE SUMMABILITY OF A DIRICHLET SERIES

By

Miss Pramila Srivastava

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1.1. Let Σa_n be a given infinite series and λ_n a positive monotonic increasing function of n, tending to infinity with n. Then

$$(*) \qquad A_\lambda^k(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^k a_n, \; k \geqslant 0,$$

defines the Riesz sum of type λ and order k for the series Σa_n. If

$$\lim_{\omega \to \infty} \omega^{-k} A_\lambda^k(\omega) = s,$$

the series Σa_n is said to be summable (R, λ, k) to the sum s.[1] And if $\omega^{-k} A_\lambda^k(\omega)$ is of bounded variation in (h, ∞), where h is a finite positive number, then Σa_n is said to be absolutely summable (R, λ, k), or summable $|R, \lambda, k|$.[2]

Having extended the definition (*) to the case $k > -1$,[3] we denote

$$k \int^\omega |A_\lambda^{k-1}(x)| \, dx,$$

or

$$\int^\omega |dA_\lambda^k(x)|,$$

where $k > 0$, by $A_\lambda^k[\omega]$. The lower limit of the above integrals may be any finite positive number. Here and elsewhere in such cases we omit the lower limit.

1.2. The properties of σ_k, the abscissa of summability (R, λ, k) of the Dirichlet series $\Sigma a_n \exp(-\lambda_n s)$ considered as a function of k were first inves-

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1. Riesz (11); Hardy and Riesz (8); Chandrasekharan and Minakshisundaram (6).

2. Obrechkoff (9), (10); Chandrasekharan and Minakshisundaram (6).

3. See Srivastava (13).

tigated by Bohr.[1] He proved that in the case of an ordinary Dirichlet series, i. e. for $\lambda_n = \log n$, and integral k,

$$\sigma_k - \sigma_{k+1} \geqslant \sigma_{k+1} - \sigma_{k+2}.^{2}$$

This result was later extended by Riesz[3] so as to apply to the general Dirichlet series and to all orders of summation, integral or non-integral. He showed that in all cases σ_k is a convex function of k.

Summability $|R, \lambda, k|$ was introduced and applied to the summation of general Dirichlet series by Obrechkoff.[4] He established the existence of the abscissa g_k of summability $|R, \lambda, k|$ of the series $\Sigma a_n exp(-\lambda_n s)$ and evaluated it. Subsequent to his investigations the properties of g_k, mostly its relationship with σ_k, have been considered by a number of authors.[5] The object of the present paper is to show that, in the case of absolute summability also, the abscissa g_k continues to be a convex function of k.

I take this opportunity to express my deep sense of gratitude to Prof. B. N. Prasad for his helpful guidance in the preparation of this paper.

2.1. THEOREM. *If* g_k, *the abscissa of summability* $|R, \lambda, k|$ *of the series* $\Sigma a_n exp(-\lambda_n s)$ *is finite, then it is a convex function of* k. *That is, for* $0 < k \leqslant p \leqslant r$, we have

$$g_p \leqslant \frac{(p-k) g_r + (r-p) g_k}{(r-k)}.$$

2.2. We require the following lemmas for the proof of our theorem.

LEMMA 1. *If* $k+\delta \geqslant 0$, *then for all* x *and if* $0 > k+\delta > -1$, *then for almost all* x

$$A_\lambda^{k+\delta}(x) = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} A_\lambda^k(t)\,dt,$$

where $k > -1$ *and* $\delta > 0$.

1. Bohr (2), (3).
2. See also Hardy and Littlewood (7, Theorem 21).
3. Riesz (12).
4. Obrechkoff (9), (10).
5. Austin (1); Borwein (4); Bosanquet (5); Tatchell (14); Chandrasekharan and Minakshisundaram (6).

The part concerning the case $k+\delta \geqslant 0$ follows from a result due to Hardy and Riesz[1] and the other part can be similarly proved.

LEMMA 2.[2] *If $k>0$ and g_k is the abscissa of summability $|R, l, k|$ for the series $\Sigma a_n l_n^{-s}$, where $l_n = \exp(\lambda_n)$, and if $g_k \geqslant 0$ then*

$$g_k = \limsup_{\omega \to \infty} \frac{A_l^k[\omega]}{\log \omega} - k.$$

LEMMA 3. *If*

$$G(x) = \int_a^x \phi(x, u)\, g'(u)\, du,$$

then

$$\int_a^\omega |dG(x)| \leqslant \overline{\underset{a<u<w}{bd}} \left\{ \phi(u, u) + \int_u^\omega |d_x \phi(x, u)| \right\} \int_a^\omega |g(u)| du.$$

With ∞ in place of ω this lemma has been given by Tatchell.[3] In the present form, also, it can be easily obtained.

LEMMA 4. *If, as $x \to \infty$,*

$$A_\lambda^k[x] = O\{V(x)\}$$

and

$$A_\lambda^r[x] = O\{W(x)\},\ k+1>r>k>0,$$

where $V(x)$, $W(x)$, $W(x)/V(x)$ and $x/\{W(x)/V(x)\}^{1/(r-k)}$ are positive monotonic increasing functions of x, then

$$A_\lambda^{k+\delta}[x] = O\{V(x)\}^{1-\delta/(r-k)} \{W(x)\}^{\delta/(r-k)},\ 0<\delta<r-k.$$

Proof. We suppose that

$$\left\{\frac{W(x)}{V(x)}\right\}^{1/(r-k)} < x,$$

for otherwise the lemma will follow from the proof given for the first theorem of consistency for strong summability (R, λ, k).[4] Denoting by C a constant which may be different at different places, we have, by

1. Hardy and Riesz (8, p. 27, Lemma 6).
2. Bosanquet (5, Theorem 34).
3. Tatchell (14, Lemma 1 (i)).
4. Srivastava (13).

Lemma 1,

$$A_\lambda^{k+\delta}[\omega]=\int^{\omega}|dA_\lambda^{k+\delta}(x)|,$$

$$=\int^{\omega}\left|d_x C\int_0^x A_\lambda^{r-1}(t)(x-t)^{k+\delta-r}\,dt\right|$$

$$=C\int^{\omega}\left|d_x\left\{\int_0^{\xi}\cdots+\int_{\xi}^{x}\cdots\right\}\right|$$

$$\leqslant C\int^{\omega}\left|d_x\int_0^{\xi}\cdots\right|+C\int^{\omega}\left|d_x\int_{\xi}^{x}\cdots\right|$$

$$=I_1(\omega)+I_2(\omega),$$

say, where

$$\xi=f(x)$$

and

$$f^{-1}(x)=x+\left\{\frac{W(x)}{V(x)}\right\}^{1/(r-k)}$$

Now, for almost all values of x,

$$\frac{d}{dx}\int_0^{\xi}A_\lambda^{r-1}(t)\,(x-t)^{k+\delta-r}\,dt$$

$$=A_\lambda^{r-1}(\xi)(x-\xi)^{k+\delta-r}\frac{d\xi}{dx}+(k+\delta-r)\int_0^{\xi}A_\lambda^{r-1}(t)\,(x-t)^{k+\delta-r-1}\,dt.$$

Therefore

$$I_1(\omega)\leqslant C\int^{\omega}|A_\lambda^{r-1}(\xi)|(x-\xi)^{k+\delta-r}\frac{d\xi}{dx}\,dx$$

$$+C\int^{\omega}dx\int_0^{\xi}|A_\lambda^{r-1}(t)|\,(x-t)^{k+\delta-r-1}\,dt$$

$$=I_{1,1}(\omega)+I_{1,2}(\omega),$$

say. And

$$I_{1,1}(\omega)\leqslant C\int_0^{f(\omega)}|A_\lambda^{r-1}(t)|\{f^{-1}(t)-t^{k+\delta-r}\,dt\}$$

$$=C\int_0^{f(\omega)}|A_\lambda^{r-1}(t)|\left\{\frac{W(t)}{V(t)}\right\}^{\{\delta/(r-k)\}-1}dt$$

$$<C\int_0^{\omega}|A_\lambda^{r-1}(t)|\left\{\frac{W(t)}{V(t)}\right\}^{\{\delta/(r-k)\}-1}dt$$

$$=O\{W(\omega)\}\left\{\frac{W(\omega)}{V(\omega)}\right\}^{\{\delta/(r-k)\}-1}+O\left[\int_0^\omega W(t)\left\{\frac{W(t)}{V(t)}\right\}^{\{\delta/(r-k)\}-2}\times\frac{d}{dt}\left\{\frac{W(t)}{V(t)}\right\}dt\right]$$

$$=O\{V(\omega)\}^{1-\delta/(r-k)}\{W(\omega)\}^{\delta/(r-k)}$$

$$+O\{V(\omega)\}\int_0^\omega\left\{\frac{W(t)}{V(t)}\right\}^{\{\delta/(r-k)\}-1}\frac{d}{dt}\left\{\frac{W(t)}{V(t)}\right\}dt$$

$$=O\{V(\omega)\}^{1-\delta/(r-k)}\{W(\omega)\}^{\delta/(r-k)}.$$

Also, changing the order of integration, we get

$$I_{1,2}(\omega)\leqslant C\int_0^{f(\omega)}|A_\lambda^{r-1}(t)|dt\int_{f^{-1}(t)}^{\omega}(x-t)^{k+\delta-r-1}dt$$

$$\leqslant C\int_0^{f(\omega)}|A_\lambda^{r-1}(t)|\{f^{-1}(t)-t\}^{k+\delta-r}dt$$

$$=O\{V(\omega)\}^{1-\delta/(r-k)}\{W(\omega)\}^{\delta/(r-k)},$$

as in case of $I_{1,1}(\omega)$. Next, we consider

$$I_2(\omega)=C\int^{w}\left|d_x\int_\xi^x A_\lambda^{r-1}(t)(x-t)^{k+\delta-r}dt\right|.$$

We have, again by Lemma 1,

$$\int_\xi^x(x-t)^{k+\delta-r}A_\lambda^{r-1}(t)\,dt$$

$$=C\int_\xi^x(x-t)^{k+\delta-r}dt\int_0^t A_\lambda^{k-1}(u)(t-u)^{r-k-1}du$$

$$=C\left\{\int_0^x dt\int_0^t\ldots du-\int_0^\xi dt\int_0^t\ldots du\right\}$$

$$=C\left\{\int_0^x du\int_u^x\ldots dt-\int_0^\xi du\int_u^\xi\ldots dt\right\}$$

$$=C\int_0^\xi A_\lambda^{k-1}(u)\,du\int_\xi^x(x-t)^{k+\delta-r}(t-u)^{r-k-1}dt$$

$$+C\int_\xi^x A_\lambda^{k-1}(u)\,du\int_u^x(x-t)^{k+\delta-r}(t-u)^{r-k-1}dt$$

$$=I_{2,1}(x)+I_{2,2}(x),$$

say.

But

$$I_{2,2}(x)=C\int_{\xi}^{x} A_\lambda^{k-1}(u)(x-u)^{\delta}\, du.$$

Therefore, for almost all values of x,

$$\left|\frac{d}{dx}I_{2,2}(x)\right|\leqslant C|A_\lambda^{k-1}(\xi)|(x-\xi)^{\delta}\frac{d\xi}{dx}+C\int_{\xi}^{x}|A_\lambda^{k-1}(u)|(x-u)^{\delta-1}du.$$

Hence

$$\int^{\omega}|d_x I_{2,2}(x)|\leqslant C\int^{\omega}|A_\lambda^{k-1}(\xi)|(x-\xi)^{\delta}\frac{d\xi}{dx}dx$$

$$+C\int^{\omega}dx\int_{\xi}^{x}|A_\lambda^{k-1}(u)|(x-u)^{\delta-1}du$$

$$=C\int^{f(\omega)}|A_\lambda^{k-1}(t)|\{f^{-1}(t)-t\}^{\delta}dt+\mathcal{J}(\omega)$$

$$=O\left\{\frac{W(\omega)}{V(\omega)}\right\}^{\delta/(r-k)}V(\omega)+\mathcal{J}(\omega),$$

where

$$\mathcal{J}(\omega)\leqslant C\int_0^{\omega}dx\int_{\xi}^{x}|A_\lambda^{k-1}(u)|(x-u)^{\delta-1}du$$

$$=C\left\{\int_0^{\omega}dx\int_0^{x}\ldots du-\int_0^{\omega}dx\int_0^{\xi}\ldots du\right\}$$

$$=C\left\{\int_0^{\omega}du\int_u^{\omega}\ldots dx-\int_0^{f(\omega)}du\int_{f^{-1}(u)}^{\omega}\ldots dx\right\}$$

$$=C\left\{\int_0^{f(\omega)}du\int_u^{f^{-1}(u)}\ldots dx+\int_{f(\omega)}^{\omega}du\int_u^{\omega}\ldots dx\right\}$$

$$=\mathcal{J}_1(\omega)+\mathcal{J}_2(\omega),$$

say. Now

$$\mathcal{J}_1(\omega)\leqslant C\int_0^{\omega}|A_\lambda^{k-1}(u)|du\int_u^{f^{-1}(u)}(x-u)^{\delta-1}dx$$

$$=C\int_0^{\omega}|A_\lambda^{k-1}(u)|\left\{\frac{W(u)}{V(u)}\right\}^{\delta/(r-k)}du$$

$$=O\{V(\omega)\}\left\{\frac{W(\omega)}{V(\omega)}\right\}^{\delta/(r-k)},$$

and

$$\begin{aligned} J_2(\omega) &= C\int_{f(\omega)}^{\omega} |A_\lambda^{k-1}(t)|(\omega-u)^\delta du \\ &= O\{\omega-f(\omega)\}^\delta V(\omega) \\ &= O\left\{\frac{W(\omega)}{V(\omega)}\right\}^{\delta/(r-k)} V(\omega). \end{aligned}$$

Thus we are left only with

$$\int^{\omega} |d_x I_{2,1}(x)| \leqslant C\int_0^{\omega} \left|d_x \int_0^{\xi} A_\lambda^{k-1}(u)\, du \int_{\xi}^{x} (x-t)^{k+\delta-r}(t-u)^{r-k-1} dt\right|$$

$$= C\int_0^{f(\omega)} \left|d_v \int_0^{v} A_\lambda^{k-1}(u)\, du \int_v^{f^{-1}(v)} \{f^{-1}(v)-t\}^{k+\delta-r}(t-u)^{r-k-1} dt\right|.$$

By Lemma 3, this gives

$$\int^{\omega} |d_x I_{2,1}(x)| \leqslant \overline{\underset{0<u<f(\omega)}{bd}}\, C\left\{\int_u^{f^{-1}(u)} [f^{-1}(u)-t]^{k+\delta-r}(t-u)^{r-k-1}dt\right.$$

$$\left. + \int_u^{f(\omega)} \left|d_v \int_v^{f^{-1}(v)} \{f^{-1}(v)-t\}^{k+\delta-r}(t-u)^{r-k-1} dt\right|\right\}$$

$$\times \int_0^{f(\omega)} |A_\lambda^{k-1}(u)|\,du$$

$$= O\left\{V(\omega)\right\} \overline{\underset{0<u<f(\omega)}{bd}}\left\{[f^{-1}(u)-u]^\delta + \int_u^{f(\omega)} \left|d_v \int_v^{f^{-1}(v)} \dots dt\right|\right\}$$

$$= O\{V(\omega)\}\left\{\frac{W(\omega)}{V(\omega)}\right\}^{\delta/(r-k)} + O\{V(\omega)\}F(\omega),$$

where

$$F(\omega) = \overline{\underset{0<u<f(\omega)}{bd}} \int_u^{f(\omega)} \left|d_v \int_v^{f^{-1}(v)} \{f^{-1}(v)-t\}^{k+\delta-r}(t-u)^{r-k-1} dt\right|.$$

Putting $t=v+\{f^{-1}(v)-v\}\tau$, we get

$$\int_v^{f^{-1}(v)} \{f^{-1}(v)-t\}^{k+\delta-r}(t-u)^{r-k-1} dt$$

$$= \int_0^1 (1-\tau)^{k+\delta-r}\{f^{-1}(v)-v\}^\delta \left\{\frac{v-u}{f^{-1}(v)-v}+\tau\right\}^{r-k-1} d\tau,$$

where both

$$\{f^{-1}(v)-v\}^\delta$$

and

$$\left\{\frac{v-u}{f^{-1}(v)-v}+\tau\right\}^{r-k-1}$$

are monotonic functions of v, the former increasing and the latter decreasing. Therefore

$$F(\omega)=\overline{\underset{0<u<f(\omega)}{bd}}\int_{u}^{f(\omega)}|d_v\int_0^1(1-\tau)^{k+\delta-r}\{f^{-1}(v)-v\}^{\delta}$$

$$\times\left\{\frac{v-u}{f^{-1}(v)-v}+\tau\right\}^{r-k-1}d\tau|$$

$$=\overline{\underset{0<u<f(\omega)}{bd}}\int_0^1(1-\tau)^{k+\delta-r}d\tau\int_u^{f(\omega)}|d_v\ldots|$$

$$\leqslant 2\int_0^1(1-\tau)^{k+\delta-r}\left\{\frac{W(\omega)}{V(\omega)}\right\}^{\delta/(r-k)}\tau^{r-k-1}d\tau$$

$$=O\left\{\frac{W(\omega)}{V(\omega)}\right\}^{\delta/(r-k)}$$

Hence

$$\int^{\omega}|d_x I_{2,1}(x)|=O\{V(\omega)\}\left\{\frac{W(\omega)}{V(\omega)}\right\}^{\delta/(r-k)},$$

which completes the proof of the lemma.

2.3. *Proof of the Theorem. Case* (i) : $r<k+1$. We suppose $g_k\geqslant 0$ and then, by Lemma 2, we have[1]

(2.3.1) $$g_k=\limsup_{\omega\to\infty}\frac{\log A_l^k[\omega]}{\log\omega}-k.$$

Therefore

$$A_l^k[\omega]=O(\omega^{k+g_k+\epsilon}),\ \epsilon>0,$$

and similarly

$$A_l^r[\omega]=O(\omega^{r+g_r+\epsilon}).$$

An application of Lemma 4 makes these two estimates together yield

$$A_l^p[\omega]=O\{\omega^{p+[g_r(p-k)+g_k(r-p)]/(r-k)}\}.$$

1. The abscissae of summabilities $|R,\lambda,k|$ and $|R,l,k|$ for the series $\Sigma a_n\,exp(-\lambda_n s)$ are the same [Tatchell (14)].

Now (2.3.1), with p in place of k, gives

$$g_p \leqslant \frac{g_r(p-k)+g_k(r-p)}{r-k}+\epsilon.$$

Making $\epsilon \to 0$, we get the required result.

Case (ii) : $r \geqslant k+1$. Let m be an integer such that

$$(r-k)/m<\tfrac{1}{2},$$

and put

$$r_q=q(r-k)/m,\ q=0,\ 1,\ 2, \ldots,\ m-1,\ m.$$

An application of the case (i) proved above gives

$$\text{(I)} \qquad g_{k+r_q} \leqslant \tfrac{1}{2}(g_{k+r_{q-1}}+g_{k+r_{q+1}}),$$

where $q=1, 2, \ldots, m-1$. Now if l be one of the numbers $1, 2, \ldots, m-1$, we write down the inequalities (I) corresponding to $q=1, 2, \ldots, l-1, l, l+1, \ldots, m-2, m-1$, multiplied respectively by $m-l$, $2(m-l)$, ..., $(l-1)\times(m-l)$, $(m-l)l, \ldots, 2l, l$. On addition these give

$$mg_{k+r_l} \leqslant (m-l)g_{r_0+k}+lg_{k+r_m},$$

or

$$g_{k+r_l} \leqslant \left(1-\frac{l}{m}\right)g_k+\frac{l}{m}g_r$$

$$=\left(1-\frac{r_l}{r-k}\right)g_k+\frac{r_l}{r-k}g_r.$$

Putting $k+r_l=p$, we get

$$g_p \leqslant \frac{(r-p)g_k+(k-p)g_r}{r-k}.$$

This completes the proof of the theorem for all values of r when $g_k \geqslant 0$. We then extend this result to the case $g_k > -M$ where M is an arbitrarily large positive number, by a change of origin.

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The University
Allahabad
India

ON THE LOGARITHMIC SUMMABILITY OF THE SUCCESSIVELY DERIVED SERIES OF A FOURIER SERIES AND OF ITS CONJUGATE SERIES

By

MISS SULAXANA KUMARI

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1.1. *Definition.* A series Σa_n is said to be summable $(R, \log w, k)$ or, in short, summable (R, k), to sum s, if

$$R_k(w) = (\log w)^{-k} \sum_{n<w} (\log \frac{w}{n})^k a_n, \; k \geqslant 0,$$

tends to a limit s, as $w \to \infty$.[1]

Let the Fourier series of a function $f(\theta)$, integrable (L) over $(-\pi, \pi)$ and periodic outside this range with period 2π, be

$$(1.11) \qquad f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n(\theta).$$

Then the series conjugate to the above Fourier series is

$$(1.12) \qquad \sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta) = \sum_{n=1}^{\infty} B_n(\theta).$$

The series

$$(1.13) \qquad \sum_{n=1}^{\infty} \frac{d^r}{d\theta^r} A_n(\theta)$$

and

$$(1.14) \qquad \sum_{n=1}^{\infty} \frac{d^r}{d\theta^r} B_n(\theta)$$

are the rth derived series of the Fourier series (1.11) and its conjugate series (1.12), respectively. Let $P(t)$ be a polynomial of $(r-1)$th degree in t;

$$(1.15) \qquad g(t) = \frac{1}{2t^r} [\{f(x+t) - P(t)\} + (-1)^r \{f(x-t) - P(-t)\}]$$

1. Riesz (7).

and

(1.16) $$h(t)=\frac{1}{2t^r}[\{f(x+t)-P(t)\}-(-1)^r\{f(x-t)-P(-t)\}],$$

$g(t)$ and $h(t)$ being defined by periodicity outside $(-\pi, \pi)$. The rth generalized derivative of $f(\theta)$, at $\theta=x$, is defined by

$$f_r(x)=\lim_{t\to 0} r!\, g(t),$$

and is equal to the rth derivative of $f(\theta)$, at $\theta=x$, in case the latter exists and is finite.[1]

In what follows we shall write

$$\phi(t)=\Phi_0(t)=\bar{\Phi}_0(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)-2s\},$$

$$\psi(t)=\tfrac{1}{2}\{f(x+t)-f(x-t)\},$$

$$\theta(t)=\frac{2}{\pi}\int_t^\infty \psi(u)\, u^{-1}\, du-s,$$

$$\Phi_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)du,\ \alpha>0,$$

$$\bar{\Phi}_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_t^\pi (\log u/t)^{\alpha-1}\phi(u)\, u^{-1}\, du,\ \alpha>0,$$

$$\phi_\alpha(t)=\Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t),$$

$$\bar{\phi}_\alpha(t)=\Gamma(\alpha+1)(\log 1/t)^{-\alpha}\bar{\Phi}_\alpha(t),$$

and define $\Psi_\alpha(t)$, $\psi_\alpha(t)$, $\bar{\Psi}_\alpha(t)$, $\bar{\psi}_\alpha(t)$, $G_\alpha(t)$, $g_\alpha(t)$, $\bar{G}_\alpha(t)$, $\bar{g}_\alpha(t)$, $H_\alpha(t)$, $h_\alpha(t)$, $\bar{H}_\alpha(t)$, $\bar{h}_\alpha(t)$, $\Theta_\alpha(t)$, $\theta_\alpha(t)$, $\bar{\Theta}_\alpha(t)$ and $\bar{\theta}_\alpha(t)$ in a similar way. We shall also make use of the following functions:

(1.17) $$L_k(t)=\int_0^1 (\log\frac{1}{u})^k \cos tu\, du,\ k>-1,$$

(1.18) $$M_k(t)=\int_0^1 (\log\frac{1}{u})^k \sin tu\, du,\ k>-1,$$

(1.19) $$=\Gamma(k+1)\sum_{p=0}^{\infty}[(-1)^p t^{2p+1}\{(2p+2)^{k+1}(2p+1)!\}^{-1}].^2$$

1. Zygmund (16).
2. Wang (10), (11).

1.2. *Introduction.* It was first proved by Zygmund[1] in 1924 that if

$$f'(x)=\lim_{t\to 0}\{f(x+t)-f(x-t)\}/2t$$

exists and is finite, then the first derived series of the Fourier series of $f(\theta)$, at $\theta=x$, is summable $(R, 1)$, to sum $f'(x)$. In 1935 Wang proved[2] that if $\{f(x+t)-f(x-t)\}/2t$ is integrable in Cauchy's sense in $(0, \delta)$ and if there is constant A such that

$$\int_0^t\left\{\frac{f(x+u)-f(x-u)}{2u}-A\right\}du=o(t),$$

as $t\to 0$, then the first derived series of the Fourier series of $f(\theta)$, at $\theta=x$, is summable $(R, 2)$ to sum A. Matsuyama proved[3] in 1950 that if

$$\frac{f(x+t)-f(x-t)}{2t}=s(R, \alpha),\ \alpha>0,$$

then (1.13), at $\theta=x$, for $r=1$, is summable $(R, \alpha+1+\delta)$, $\delta>0$, to sum s. Including and generalizing all these results, Sunouchi obtained[4] a far reaching result by proving that, if

$$\text{(i)}\quad F_x(t)=\frac{f(x+t)-f(x-t)}{2\sin\frac{t}{2}}$$

is integrable in the Cauchy sense at $t=0$, (ii) the Fourier series associated with $F_x(t)$, at $t=0$, has its (R, k) mean of order $o(\log n)$, and (iii) it is summable $(R, k+1)$ to sum s, then (1.13), at $\theta=x$, for $r=1$, is summable $(R, k+1)$, $k\geqslant 0$, to sum s.

Mohanty and Nanda[5] have proved that if

$$\int_t^\pi \frac{|F_x(u)-s|}{u}\,du=o(\log 1/t),$$

as $t\to 0$, then (1.13), at $\theta=x$, for $r=1$, is summable $(R, 1)$, to sum s.

It follows from Corollary 1 of the present paper that the above result is also included in Sunouchi's theorem as a particular case when $k=0$.

The object of the present paper is to generalize the result of Sunouchi so as to obtain the logarithmic summability of the rth derived series

1. Zygmund (14).
2. Wang (11).
3. Matsuyama (4).
4. Sunouchi (9).
5. Mohanty and Nanda (6).

of the Fourier series. We have also obtained an analogous result for the logarithmic summability of the successively derived series of the conjugate series.

As direct consequences of the combination of our theorems with some known results, we have obtained some corollaries, giving relations between (C, δ) and (R, δ) means of $g(t)$ and $h(t)$ and the $(R, r+\delta)$ means of (1.13) and (1.14), at $\theta=x$.

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2.1. We shall first prove the following theorem.

THEOREM 1.[1] *If the (R, δ) mean, for $\delta\geqslant 0$, of the Fourier series associated with $g(t)$, at $t=0$, is of order $o\{(\log w)^r\}$ and if it is summable $(R, r+\delta)$ to sum s, then the series (1.13), at $\theta=x$, is summable $(R, r+\delta)$ to sum $r!s$.*

LEMMA 1.[2] *If $R_r^k(w)$ and $R_0^k(w)$ denote the (R, k) means of the series (1.13) and of the Fourier series associated with $g(t)$, at $t=0$, respectively, then*

$$R_r^{r+\delta}(w)-r!s=o(1)+A_r\{R_0^{\delta}(w)-s\}+A_{r-1}\{R_0^{\delta+1}(w)-s\}$$
$$+\dots+A_0\{R_0^{\delta+r}(w)-s\},$$

where $A_p=(-1)^p a_p(\log w)^{-p}$ for $p=0, 1, 2, \dots r$, a_p's being constants independent of w.

LEMMA 2.[3] *Let $V(x)$ and $W(x)$ be two positive non-decreasing functions of x, defined for $x>0$. Let*

$$U_r(x)=\{V(x)\}^{1-r/k}\{W(x)\}^{r/k}.$$

Then (i) $h^k(x)=o\{W(x)\}$ *and* (ii) $h(x)=o\{V(x)\}$ *together imply that* $h^r(x)=o\{U_r(x)\}$, *for* $0<r<k$, *where* $h^r(x)$ *denotes the* (R, r) *mean of* $h(x)$.

Proof of Theorem 1. Using the notations of Lemma 1, we have

$$R_r^{r+\delta}(w)-r!s=o(1)+A_r\{R_0^{\delta}(w)-s\}+A_{r-1}\{R_0^{\delta+1}(w)-s\}$$

(2.11) $$+\dots+A_0\{R_0^{\delta+r}(w)-s\},\ \delta\geqslant 0,$$

where $A_p=o\{(\log w)^{-p}\}$, $p=0, 1, 2, \dots r$. Now, since by hypothesis

1. Our theorem shows that the conditions satisfied in Theorem 2 of a recently published paper of Misra (5) are sufficient, but *not necessary*.
2. Misra (5).
3. Riesz (8).

$$R_0^{\delta}(w)-s=c\{(\log w)^r\} \quad \text{and} \quad R_0^{\delta+r}(w)-s=o(1),$$

it follows from Lemma 2 that

$$R_0^{\delta+1}(w)-s=o\{\log w)^{r-1}\},$$

$$R_0^{\delta+2}(w)-s=o\{(\log w)^{r-2}\},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$R_0^{\delta+r-1}(w)-s=o(\log w).$$

Substitution of the above estimations in (2.11) yields

$$R_r^{\delta+r}(w)-r!s=o(1),$$

which proves Theorem 1.

2.2. Corollary 1. *If, for integral values of* $\delta \geqslant 0$,

$$\int_t^{\pi} |J_\delta(u)|u^{-1}\,du=o\{(\log 1/t)^{\delta+1}\}, \text{ as } t\to 0, \text{ for } \delta\geqslant 0,\ r=1,$$

where

$$J_\delta(t)=\frac{1}{\Gamma(\delta)}\int_t^{\pi}\left(\log\frac{u}{t}\right)^{\delta-1}\{g(u)-s\}\,du, \text{ for } \delta>0,\ r=1,$$

$$J_0(t)=g(t)-s,\ r=1,$$

then the series (1.13), *at* $\theta=x$, *for* $r=1$, *is summable* $(R,\ 1+\delta)$, *to sum* s.

The above corollary is obtained directly from Theorem 1 with the help of Lemmas 3 and 5 given below.

Lemma 3. *If, for integral values of* $\delta\geqslant 0$,

$$\int_t^{\pi} |\bar{\Phi}_\delta(t)|\,t^{-1}\,dt=o\left\{\left(\log\frac{1}{t}\right)^{\delta+1}\right\},$$

as $t\to 0$, *then the Fourier series* (1.11), *at* $\theta=x$, *is summable* $(R,\ \delta+1)$, *to sum* s.

The above lemma for $\delta=0$ follows from a result of Hardy,[1] and for integral values of $\delta>0$ from a result of Wang.[2]

Lemma 4.[3] *If* $k\geqslant\delta>0$, *then*

1. Hardy (1).
2. Wang (10).
3. Wang (12).

$$R_w{}^k - s = o\,(1) + \frac{2w\,(\log w)^{-k}}{\pi}\,\frac{\Gamma(k+1)}{\Gamma(k+1-\delta)}\int_0^{\pi} L_{k-\delta}(wu)\overline{\Phi}_{\delta}(u)\,du,$$

where $R_w{}^k$ denotes the (R, k) mean of (1.11), *at $\theta = x$.*

LEMMA 5. *If*

$$\int_t^{\pi} |\overline{\Phi}_{\delta}(u)|\,u^{-1}\,du = o\,\{(\log \tfrac{1}{t})^{\delta+p+1}\},$$

as $t \to 0$, for $-1 < p < \infty$, $\delta > 0$, then $R_w{}^{\delta} - s = o\{(\log w)^{p+1}\}$, as $w \to \infty$, $R_w{}^{\delta}$ being the (R, δ) mean of (1.11), *at $\theta = x$.*

Proof. Putting $k = \delta$ in Lemma 4 we have

$$R_w{}^{\delta} - s = o(1) + O\left\{w(\log w)^{-\delta}\left|\int_0^{\pi}\overline{\Phi}_{\delta}(u)L_0(wu)\,du\right|\right\}$$

$$= o(1) + O\left\{w(\log w)^{-\delta}\int_0^{w^{-1}} u|\overline{\Phi}_{\delta}(u)|u^{-1}\,du\right\}$$

$$+ O\left\{w(\log w)^{-\delta}\left|\int_{w^{-1}}^{\pi}\overline{\Phi}_{\delta}\,(u)\,(wu)^{-1}du\right|\right\}$$

$$= o(\log w)^{p+1}\} + O\{w(\log w)^{-\delta}\left(v\int_v^{\pi}|\overline{\Phi}_{\delta}\,(u)|u^{-1}\,du\right)_{v=0}^{w^{-1}}\}$$

$$+ O\left\{w(\log w)^{-\delta}\int_0^{w^{-1}} dv \int_v^{\pi}|\overline{\Phi}_{\delta}\,(u)|u^{-1}\,du\right\}$$

$$= o\{(\log w)^{p+1}\},$$

which gives the lemma.

For the case when δ is not necessarily an integer and $r > 1$, we have the following result.

COROLLARY 1. *If*

(i) $$\int_t^{\pi} |J_{\delta}\,(u)|u^{-1}\,du = o\{(\log \tfrac{1}{t})^{r+\delta}\},$$

as $t \to 0$, for $\delta \geqslant 0$, where $J_{\delta}\,(u)$ is defined as in Corollary 1 with the difference that r is supposed to be $\geqslant 1$, and (ii)[1] *the Fourier series associated with $g(t)$, at $t = 0$, is summable $(R, r+\delta)$ to sum s, then* (1.13), *at $\theta = x$, is summable $(R, r+\delta)$ to sum $r!s$.*

1. For the case $\delta = 0$, $r = 1$, this condition becomes redundant.

2.3. COROLLARY 2. *If*

$$\int_t^{\pi} |g_{r+\delta}(u)-s|u^{-1}\,du=o\{(\log \frac{1}{t})^{r+\delta}\},\ \delta\geqslant 0,$$

and the Fourier series associated with $g(t)$, at $t=0$, is summable in some Cesàro sense to sum s, and further the (R, δ) mean of the Fourier series associated with $g(t)$ at $t=0$, is of order $o\{(\log w)^r\}$, then the series (1.13), at $\theta=x$, is summable $(R, r+\delta)$ to sum $r!s$.

In particular, if

$$\int_t^{\pi} |g_{\delta}(u)-s|u^{-1}\,du=o\{(\log\frac{1}{t})^{r+\delta}\},$$

as $t\to 0$, for $\delta\geqslant 0$, and the Fourier series associated with $g(t)$, at $t=0$, is summable in some Cesàro sense, to sum s, then the series (1.13), at $\theta=x$, is summable $(R, r+\delta)$ $\delta\geqslant 0$, to sum $r!s$.

The above result can be obtained directly from Theorem 1 with the help of the following lemmas.

LEMMA 6.[1] *If, for $-1<p<\infty$, $\delta\geqslant 0$,*

$$\int_t^{\pi} |\phi_{\delta}(u)|u^{-1}\,du=o\{(\log\frac{1}{t})^{p+1}\},$$

as $t\to 0$, then $C_{\delta}(w)=o\{(\log w)^{p+1}\}$, as $w\to\infty$, where $C_{\delta}(w)$ denotes the (R, w, δ) mean of (1.11), at $\theta=x$.

LEMMA 7.[2] *If $\Sigma a_n - s=o(1)(C, k)$, for some $k>0$, and $C_{\alpha}(w)-s=o\{(p(w))^{\alpha}\}$, where $p(w)\to\infty$ as $w\to\infty$ and $C_{\alpha}(w)$ denotes the (R, w, α) mean of Σa_n, then $\Sigma a_n - s=o(1)(R, \mu(w), \alpha)$, where*

$$\mu(w)=\text{Exp}\left\{\int^{w} \frac{dt}{t\,p(t)}\right\}.$$

3.1. We consider in this section the (R, k) summability of (1.14). We shall prove the following theorem.

THEOREM 2. *If the conjugate series corresponding to $h(t)$, at $t=0$, is summable $(R, r+\delta)$ to sum s and if its (R, δ) mean is of order $o\{(\log w)^r\}$, then the series (1.14), at $\theta=x$, is summable $(R, r+\delta)$, to sum $r!s$, where $\delta\geqslant 0$.*

The following lemmas will be required for the proof of Theorem 2.

1. Sulaxana Kumari (3).
2. Zygmund (15).

LEMMA 8.

(3.11) $$\frac{d}{dt}\{tM_k(t)\}=kM_{k-1}(t) \text{ for } k>0;$$

(3.12) $$M_k(t)=O(1) \text{ for } k\geqslant 0,\ t\geqslant 0;$$

(3.13) $$M_k(t)=O\left\{t^{-1}\left(\log\frac{1}{t}\right)^k\right\} \text{ for } k\geqslant 0,\ t>2;$$

(3.14) $$\frac{2}{\pi}\int_0^\infty M_k(t)\sin xt\,dt=\begin{cases}(\log\frac{1}{x})^k & \text{for } 0<x<1\\ 0 & \text{for } x\geqslant 1;\end{cases}$$

(3.15) $$M_k^r(0)=\begin{cases}0 & \text{for } r \text{ even,}\\ O(1) & \text{for } r \text{ odd};\end{cases}$$

(3.16) $$M_k^r(t)=O(t^{-r}), \text{ for } k\geqslant r\geqslant 1,\ t\geqslant 0;$$

(3.17) $$t^r M_k^r(t)=\sum_{p=0}^{r} a_p M_{k-p}, \text{ for } k\geqslant r>1,\ t\geqslant 0,$$

a_p's being constants such that

$$a_0=(-1)^r r!.$$

Proof. The results (3.11)...(3.14) are due to Wang.[1] (3.15) follows easily from (1.19). Hence we give here the proofs of (3.16) and (3.17). Since for $k>0$,

$$\frac{d}{dt}\{tM_k(t)\}=kM_{k-1}(t),$$

we have

$$t^2M_k'(t)=ktM_{k-1}(t)-tM_k(t).$$

Differentiating again, we get

$$t^2M_k''(t)=2M_k(t)-3kM_{k-1}(t)+k(k-1)M_{k-2}(t).$$

Hence

$$M_k''(t)=O(t^{-2}), \text{ for } k\geqslant 2,\ t\geqslant 0.$$

1. Wang (13).

Proceeding similarly, we obtain

$$M_k^r(t)=t^{-r}\sum_{p=0}^{r} M_{k-p}(t)\cdot a_p, \text{ for } k\geqslant r\geqslant 1,\ t\geqslant 0,$$

$$=O(t^{-r}),$$

where it is easy to see that $a_0=(-1)^r r!$ and $a_r=k(k-1)\dots(k-r+1)$.

LEMMA 9.[1] *If $\bar{B}^{\alpha}(w)$ denotes the (R,α) mean of the conjugate series corresponding to $h(t)$, at $t=0$, then*

$$\bar{B}^{\alpha}(w)=\frac{2w}{\pi(\log w)^{\alpha}}\int_0^{\infty} h(t)\,M_{\alpha}(wt)\,dt,\ \alpha>0.$$

3.2. *Proof of Theorem* 2. We have

$$\sum_{n=1}^{\infty}\frac{d^r}{d\theta^r}\{B_n(\theta)\}=\begin{cases}(-1)^{(r+1)/2}\sum_{n=1}^{\infty} n^r A_n(\theta), \text{ for } r \text{ odd},\\ (-1)^{r/2}\sum_{n=1}^{\infty} n^r B_n(\theta), \text{ for } r \text{ even}.\end{cases}$$

Let

$$R_{2p+1}{}^k(w)=\frac{1}{(\log w)^k}\sum_{n<w}\left(\log\frac{w}{n}\right)^k\frac{d^{2p+1}}{d\theta^{2p+1}}\{B_n(\theta)\}$$

$$=(-1)^{p+1}(\log w)^{-k}\sum_{n<w}\left(\log\frac{w}{n}\right)^k n^{2p+1}A_n(\theta)$$

and

$$R_{2p}{}^k(w)=(-1)^p(\log w)^{-k}\sum_{n<w}\left(\log\frac{w}{n}\right)^k n^{2p}B_n(\theta),$$

where $k\geqslant 2p+1$ in the former case and $k\geqslant 2p$ in the latter case. Write

$$\eta(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)\}\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}A_n(x)\cos nt,$$

$$\psi(t)=\tfrac{1}{2}\{f(x+t)-f(x-t)\}\sim\sum_{n=1}^{\infty}B_n(x)\sin nt,$$

$$\xi(t)=\tfrac{1}{2}\{f(x+t)+(-1)^{r+1}f(x-t)\}.$$

1. Wang (13).

Now integration by parts yields

$$\int_0^\infty M_k^{2p+1}(wt)\cos nt\,dt = \frac{n}{w}\int_0^\infty M_k^{2p}(wt)\sin nt\,dt$$

$$= \dots \quad \dots \quad \dots$$

(3.21) $$=(-1)^p (n/w)^{2p+1}\int_0^\infty M_k(wt)\sin nt\,dt.$$

Similarly

(3.22) $$\int_0^\infty M_k^{2p}(wt)\sin nt\,dt = (-1)^p (n/w)^{2p}\int_0^\infty M_k(wt)\sin nt\,dt.$$

Consequently it follows from (3.14), (3.21) and (3.22) that

$$(-1)^{p+1}\int_0^\infty M_k^{2p+1}(wt)\cos nt\,dt = \begin{cases} (-1)^{2p+1}\tfrac{1}{2}\pi n^{2p+1} w^{-2p-2}\left(\log\frac{w}{n}\right)^k, & \text{for } n<w \\ 0, \text{ for } n\geqslant w. \end{cases}$$

and

$$(-1)^p\int_0^\infty M_k^{2p}(wt)\sin nt\,dt = \begin{cases} (-1)^p\tfrac{1}{2}\pi n^{2p} w^{-2p-1}\left(\log\frac{w}{n}\right)^k, \text{ for } n<w, \\ 0, \text{ for } n\geqslant w. \end{cases}$$

Hence

$$R_{2p+1}^k(w) = (-1)^{p+1}(\log w)^{-k}\sum_{n<w}\left(\log\frac{w}{n}\right)^k n^{2p+1}A_n(x)$$

$$= -\frac{2}{\pi}w^{2p+2}(\log w)^{-k}\sum_{n<w}A_n(x)\int_0^\infty M_k^{2p+1}(wt)\cos nt\,dt$$

and

$$R_{2p}^k(w) = \frac{2}{\pi}w^{2p+1}(\log w)^k\sum_{n<w}B_n(x)\int_0^\infty M_k^{2p}(wt)\sin nt\,dt..$$

This gives finally

(3·23) $$R_r^k(w) = (-1)^r\frac{2}{\pi}w^{r+1}(\log w)^{-k}\int_0^\infty M_k^r(wt)\,\xi(t)\,dt.$$

Thus the necessary and sufficient condition that the rth derived series of the conjugate series of the Fourier series of $f(\theta)$, at $\theta=x$, be summable (R, k), for $k\geqslant r$, is that

$$\lim_{w\to\infty} w^{r+1} (\log w)^{-k} \int_0^\infty M_k^r (wt)\, \xi(t)\, dt$$

exists.

Now, since $M_k^r(-wt)=(-1)^{r+1} M_k^r(wt)$, we have

$$(-1)^{r+1} \int_{-\infty}^0 M_k^r (wt) f(\theta+t)\, dt = (-1)^{r+2} \int_\infty^0 M_k^r(-wt) f(\theta-t)\, dt$$

$$= \int_0^\infty M_k^r(wt) f(\theta-t)\, dt.$$

Hence

$$(3.24) \quad \int_0^\infty M_k^r (wt)\{f(\theta+t)+(-1)^{r+1} f(\theta-t)\}\, dt = \int_{-\infty}^\infty M_k^r(wt) f(\theta+t)\, dt.$$

Next, it follows from (3.13), (3.17) in Lemma 8, that

$$w^{r+1} (\log w)^{-k} \int_0^\infty t^p M_k^r (wt)\, dt = w^{r+1} (\log w)^{-k} \Big\{ \{ t^p w^{-1} M_k^{r-1}(wt)$$

$$-p w^{-2} t^{p-1} M_k^{r-2}(wt) + p(p-1)\, w^{-3} t^{2p-2} M_k^{r-3}(wt) + \ldots$$

$$+(-1)^{p-1} p!\, t\, w^{-p} M_k^{r-p}(wt) \Big\}_{t=0}^{\infty}$$

$$+(-1)^p w^{r+1} (\log w)^{-k} p!\, w^{-p} \int_0^\infty M_k^{r-p}(wt)\, dt$$

$$= O\Big\{ (\log wt)^k (\log w)^{-k} t^{p-r} \Big\}_{t=\infty}$$

$$+(-1)^p w^{r-p} (\log w)^{-k} \Big\{ M_k^{r-p-1}(wt) \Big\}_{t=0}^{\infty}.$$

Now, if $(r-p)$ is an odd integer, then $M_k^{r-p-1}(t)$ vanishes for $t=0$, and for such value

$$w^{r+1} (\log w)^{-k} \int_0^\infty t^p M_k^r (wt)\, dt = O\Big\{ (\log wt)^k (\log w)^{-k} t^{p-r} \Big\}_{t=\infty}$$

$$(3.25) \qquad = o(1) \text{ for } p<r.$$

Again

$$w^{r+1} (\log w)^{-k} \int_{-\infty}^\infty t^p M_k^r (wt)\, dt$$

$$=w^{r+1}(\log w)^{-k}\left\{\int_0^\infty t^p M_k^r(wt)\,dt-\int_\infty^0(-1)^p t^p(-1)^{r+1}M_k^r(wt)\,dt\right\}$$

$$=w^{r+1}(\log w)^{-k}\left\{\int_0^\infty t^p M_k^r(wt)\,dt+(-1)^{r+p+1}\int_0^\infty t^p M_k^r(wt)\,dt\right\}$$

$$=\begin{cases}0, \text{ for } r-p \text{ even}\\ 2w^{r+1}(\log w)^{-k}\int_0^\infty M_k^r(wt)\,t^p\,dt \text{ for } {}^\circ r-p \text{ odd.}\end{cases}$$

(3.26) $\quad =0$

for $p<r$, by (3.25).

Thus from (3.23), (3.24), and (3.26) we obtain

$$R_r^k(w)=\frac{(-1)^r}{\pi}w^{r+1}(\log w)^{-k}\int_{-\infty}^\infty M_k^r(wt)\{f(\theta+t)-P(t)\}\,dt$$

$$=(-1)^r w^{r+1}(\log w)^{-k}\pi^{-1}\int_0^\infty M_k^r(wt)[\{f(\theta+t)-P(t)\}$$
$$+(-1)^{r+1}\{f(\theta+t)-P(-t)\}]\,dt$$

(3·27) $$=(-1)^r\frac{2}{\pi}\,w^{r+1}(\log w)^{-k}\int_0^\infty h(t)\,t^r M_k^r(wt)\,dt$$

(3·28) $$=(-1)^r\frac{2w}{\pi}(\log w)^{-k}\int_0^\infty h(t)\left\{\sum_{p=0}^{r}a_p M_{k-p}(wt)\right\}dt$$

by (3.17), where a_p's are defined as in Lemma 8.

Now by Lemma 9 and (3.28) we have

$$R_r^{r+\delta}(w)=(-1)^r\sum_{p=0}^{r}a_p(\log w)^{-p}\bar{B}^{r+\delta-p}(w)$$

$$=(-1)^r\{a_0\bar{B}^{r+\delta}(w)+a_1\bar{B}^{r+\delta-1}(w)/(\log w)$$
(3.29) $$+a_2\bar{B}^{r+\delta-2}(w)(\log w)^{-2}+\ldots+a_r\bar{B}^{\delta}(w)(\log w)^{-r}\},$$

where $\bar{B}^k(w)$ denotes the (R, k) mean of the conjugate series corresponding to $h(t)$, at $t=0$. Now, since by hypothesis $\bar{B}^\delta(w)=o\{(\log w)^r\}$ and $\bar{B}^{r+\delta}(w)=s$, it follows by Lemma 2 that $B^{\delta+q}(w)=o\{(\log w)^{r-q}\}$ for $q=1, 2, \ldots (r-1)$. Thus, since $a_0=(-1)^r r!$, we have from (3.22)·

$$R_r^{r+\delta}(w)=r!s+o(1),$$

which completes the proof of Theorem 2.

3.3. Corollary 3. *If, for* $\delta \geqslant 0$,

$$\text{(i)} \quad \int_t^\pi |I_\delta(u)| u^{-1} du = o\left\{\left(\log \frac{1}{t}\right)^r\right\},$$

as $t \to 0$, *where*

$$I_0(t) = I(t) = \frac{2}{\pi}\int_t^\infty h(u) \,.\, u^{-1}\, du - s,$$

and

$$I_\delta(t) = \delta\left(\log \frac{1}{t}\right)^{-\delta} \int_t^\pi \left(\log \frac{u}{t}\right)^{\delta-1} I(u)\, du, \ \delta > 0,$$

and if

$$\text{(ii)} \quad \lim_{t\to 0} \frac{2}{\pi}\int_t^\infty h(u) \,.\, u^{-1}\, du \to s(R, r+\delta),$$

then the series (1.14), *at* $\theta = x$, *is summable* $(R, r+\delta+1)$, *to sum rls.*

In order to obtain the above result, we make use of the following lemmas.

Lemma 10.[1] *If* $a \geqslant 0$, *then*

$$\bar{R}_w{}^{\delta+1} - s = \frac{2}{\pi}\Gamma(a+1) w(\log w)^{-\alpha-1} \int_0^\pi \bar{\theta}_\alpha(u) M_0(wu) du + o(1),$$

where $\bar{R}_w{}^{\delta+1}$ *denotes the* $(R, a+1)$ *mean of* (1.12), *at* $\theta = x$.

Lemma 11. *If, for* $-1 < p < \infty$, $\delta \geqslant 0$,

$$\int_t^\pi |\bar{\theta}_\delta(u)|/u\, du = o\left\{\left(\log \frac{1}{t}\right)^{p+1}\right\},$$

as $t \to 0$, *then*

$$\bar{R}_w{}^{\delta+1} - s = o\{(\log w)^{p+1}\},$$

where $\bar{R}_w{}^{\delta+1}$ *denotes the* $(R, \delta+1)$ *mean of* (1.12), *at* $\theta = x$.

Proof. Since $M_0(wu) = (1 - \cos wu)/(wu)$, it follows from Lemma 10 that

$$\bar{R}_w{}^{\delta+1} - s = O\left\{w(\log w)^{-\delta-1} \int_0^{w^{-1}} |\theta_\delta(u)| du\right\}$$

$$+ O\left\{w(\log w)^{-\delta-1} \int_{w^{-1}}^\pi |\bar{\theta}_\delta(u)| (wu)^{-1} du\right\}$$

1. Wang (13).

$$=o\{\log w)^{p+1}\},$$

as in the proof of Lemma 6.

LEMMA 12.[1] *If* $\delta \geqslant 0$ *and* $\bar{\theta}_\delta(t)=o(1)$, *as* $t \to 0$, *then* (1.12), *at* $\theta=x$, *is summable* $(R, \delta+1)$ *to sum* s.

It follows from Lemma 10 and the condition (i) of Corollary 3, that the $(R, \delta+1)$ mean of the conjugate series corresponding to $h(t)$, at $t=0$, is of order $o\{(\log w)^r\}$. Also Lemma 12 and condition (ii) of Corollary 3 yield the convergence of the $(R, r+\delta+1)$ mean of the conjugate series corresponding to $h(t)$, at $t=0$, to sum A. Hence, all the conditions of Theorem 2 being fulfilled, Corollary 3 is obtained easily.

3.4. COROLLARY 4. *If, for* $\delta \geqslant 0$,

$$\int_t^\pi |h_{r+\delta}(u)| u^{-1}\, du = o\left\{\left(\log \frac{1}{t}\right)^{r+\delta}\right\},$$

as $t \to 0$, *the conjugate series corresponding to* $h(t)$, *at* $t=0$, *is summable in some Cesàro sense to sum* s, *and further the* (R, δ) *mean of the conjugate series corresponding to* $h(t)$, *at* $t=0$, *is of order* $o\{(\log w)^r\}$, *then the series* (1.14), *at* $\theta=x$, *is summable* $(R, r+\delta)$ *to sum* $r!\,s$.

In particular, if for $\delta \geqslant 0$,

$$\int_t^\pi |h_\delta(u)| u^{-1}\, du = o\left\{\left(\log \frac{1}{t}\right)^{r+\delta}\right\},$$

as $t \to 0$, *and if the conjugate series corresponding to* $h(t)$, *at* $t=0$, *is summable in some Cesàro sense to sum* s, *then the series* (1.14), *at* $\theta=x$, *is summable* $(R, r+\delta)$ *to sum* $r!\,s$.

The above result can be obtained directly from Theorem 2 with the help of Lemma 7 and the following lemma.

LEMMA 13.[2] *If for* $\delta \geqslant 0$, $-1<p<\infty$,

$$\int_t^\pi |\psi_\delta(u)| u^{-1}\, du = o\left\{\left(\log \frac{1}{t}\right)^{p+1}\right\},$$

as $t \to 0$, *then* $C_\delta(w)=o\{(\log w)^{p+1}\}$ *as* $w \to \infty$, *where* $C_\delta(w)$ *denotes the* (R, w, δ) *mean of* (1.12), *at* $\theta=x$.

1. Wang (13).
2. Sulaxana Kumari (2).

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